Riesz transform and perturbation

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1 Introduction

Let M be a complete non-compact Riemannian manifold, ∇ its Riemannian gradient, μ the Riemannian volume, Δ the (non-negative) Laplace-Beltrami operator. For $p \in (1, +\infty)$, one says that the Riesz transform $\nabla\Delta^{-1/2}$ is bounded on $L^p(M, \mu)$, or that (R_p) holds on M, if

$$\|\nabla f\|_p \leq C\|\Delta^{1/2} f\|_p, \ \forall f \in C_0^\infty(M).$$

It is easy to see that (R_2) always holds, but the study of (R_p) for $p \neq 2$ has given rise to many works in the last two decades. It has recently appeared that:

-(R_p) may or may not hold, depending on the geometry of M, but also on p ; in particular, (R_p) is more difficult to obtain for $p > 2$.

-one can give sufficient conditions for (R_p) in terms of heat kernel estimates, essentially in the class of manifolds satisfying the volume doubling property.

For details, see [8, 3, 2], and for more results on (R_p), see [6, 9, 10, 12, 16].

The sufficient conditions for (R_p) when $1 < p < 2$ which were given in [8] are invariant under quasi-isometry, but it is unknown whether (R_p) is invariant under quasi-isometry in this range. For $p > 2$, it is unlikely that (R_p) may be invariant under quasi-isometry, but one feels that the conditions given in [3] may be compatible with milder perturbations of the metric.

Indeed, another sufficient condition, which was first given in the framework of divergence form second order differential operators (see [17]), but which may be extended to the Riemannian setting (see [2]) can be seen to be stable under perturbation ([4]). However, this condition is only valid inside the class of manifolds satisfying upper and lower Gaussian estimates for the heat kernel, that is, in particular, the volume doubling property.

In the present work, we show that (R_p) is stable under a bounded perturbation of the metric with some L^q-integrability, under some local assumption and a mild ultracontractivity of the heat semigroup. The latter conditions hold on manifolds with bounded geometry. Note that we make no volume growth assumption whatsoever.

Note also that we do not have to assume that some norm of the perturbation is small; compare with [11, Theorem 3.1].

The plan of the paper is as follows. We start in Section 2 with the framework of second-order elliptic differential operators in divergence form in the Euclidean space. Here the main idea is introduced. In Section 3, a modification is made in order to treat the case of weighted operators. This opens the way to the treatment of the Riemannian case in Section 4. Our methods also work in a discrete setting of Markov chains on graphs, but we will not pursue this here. See however [15] for another result in this direction.

2 The Euclidean case

Let

$$H = -\operatorname{div} A\nabla$$

be a divergence-form operator in $\mathbb{R}^D$.

Here A is a measurable function on $\mathbb{R}^D$ with values in the real symmetric $D \times D$ matrices, and is assumed to satisfy an ellipticity estimate

$$C^{-1} \leq A(x) \leq C$$

for some $C > 0$ and all $x \in \mathbb{R}^D$. The precise definition of H is via the usual quadratic form method.

The Riesz transform associated with H is the operator $\nabla H^{-1/2}$. It was proved in [5] that such an operator is bounded on L^p for $1 < p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(n, C)$, and that without any further assumption this range of exponents is sharp. A necessary condition for the Riesz transform to be bounded for some $p > 2$ was subsequently given in [17]. For more on this, see also the survey [1].

Let now

$$H = -\operatorname{div} A\nabla \quad , \quad H_0 = -\operatorname{div} A_0\nabla$$

be two divergence-form operators in $\mathbb{R}^D$, satisfying the same conditions. We regard H as a perturbation of H_0 : formally

$$H = H_0 + \operatorname{div} a \nabla$$

where we have set

$$a(x) = A_0(x) - A(x), \quad x \in \mathbb{R}^D \quad .$$

Let us now assume that $a \in L^q$ (thus $a \in L^q \cap L^\infty$), for some $q \in [1, +\infty)$. The point of this condition is that then the operator of pointwise multiplication by a is bounded from L^r to L^s if $1 \leq s \leq r \leq +\infty$ and $s^{-1} - r^{-1} \leq q^{-1}$; this follows from Hölder's inequality $\|aF\|_s \leq \|a\|_w \|F\|_r$ for $F\colon \mathbb{R}^D \to \mathbb{R}^D$ and $s^{-1} - r^{-1} = w^{-1}$.

Here is our main result in this setting, which we shall use later as a basis for further extensions.

Theorem 2.1 *Let $p_0 > 2$. Assume that the global Riesz transform $\nabla H_0^{-1/2}$ of the reference operator and the local Riesz transform $\nabla(I+H)^{-1/2}$ of the perturbed operator are bounded on L^p for all $p \in (2, p_0)$. Then the global Riesz transform $\nabla H^{-1/2}$ of the perturbed operator is bounded in L^p for all $p \in (2, p_0)$.*

Theorem 2.1 will be a consequence of the following Berry-Esseen type estimate for the difference of resolvents. Let e^{-tH_0} and e^{-tH} be the heat semigroups associated with H_0 and H.

Proposition 2.2 *Let $2 < p < p_1$. Assume that*

$$\|\nabla e^{-tH_0}\|_{p_1 \to p_1} \leq C t^{-1/2}, \ \forall\, t > 0, \tag{1}$$

and that the local Riesz transform $\nabla(I+H)^{-1/2}$ is bounded in L^p. Then there exists $\varepsilon > 0$ such that

$$\|\nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1}\|_{p \to p} \leq C t^{-\varepsilon-(1/2)}, \tag{2}$$

for $t \geq 1$. *In addition,*

$$\|\nabla(I+tH)^{-1}\|_{p\to p} \leq Ct^{-1/2} \quad \text{and } \|\nabla e^{-tH}\|_{p\to p} \leq Ct^{-1/2} \tag{3}$$

for $t > 0$.

Theorem 2.1 follows from Proposition 2.2 directly by using the formula

$$H^{-1/2} = \frac{1}{\pi}\int_0^{+\infty} (I+tH)^{-1} t^{-1/2}\, dt. \tag{4}$$

Indeed, let $2 < p < p_1 < p_0$. Assumption (1) follows from the assumed boundedness of $\nabla H_0^{-1/2}$ on L^{p_1} and the universal estimate

$$\|H_0^{1/2} e^{-tH_0}\|_{p_1\to p_1} \leq Ct^{-1/2}, \ \forall t > 0,$$

a consequence of the analyticity of e^{-tH_0} on L^{p_1}. Then using formula (4) one may write

$$\nabla H^{-1/2} - \nabla H_0^{-1/2} = \frac{1}{\pi}\int_0^{+\infty} \left(\nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1}\right) t^{-1/2}\, dt.$$

Now the operator

$$\int_1^{+\infty} \left(\nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1}\right) t^{-1/2}\, dt$$

is clearly bounded on L^p thanks to the estimate (2), and the two operators $\int_0^1 \nabla(I+tH)^{-1} t^{-1/2}\, dt$ and $\int_0^1 \nabla(I+tH_0)^{-1} t^{-1/2}\, dt$ differ from the local Riesz transforms $\nabla(I+H)^{-1/2}$ and $\nabla(I+H_0)^{-1/2}$ by an operator which is obviously bounded on L^p. Note that since $H_0^{1/2}(I+H_0)^{-1/2}$ is bounded on L^p because of the analyticity of e^{-tH_0} on L^p (a kind of argument we will use very often in the sequel without extra fanfare), the boundedness of $\nabla(I+H_0)^{-1/2}$ follows from that of $\nabla H_0^{-1/2}$.

More precisely, applying (4) to $I+H$ instead of H and changing variables, one finds

$$\begin{aligned} &\int_0^1 \nabla(I+tH)^{-1} t^{-1/2}\, dt - \pi\nabla(I+H)^{-1/2} \\ = &\int_0^1 \nabla(I+tH)^{-1} \left(t^{-1/2} - t^{-1/2}(1-t)^{-1/2}\right)\, dt, \end{aligned}$$

which one sees to be bounded on L^p given the first estimate in (3).

A less direct route to deduce Theorem 2.1 from Proposition 2.2 consists in using the second estimate in (3) together with the main result in [3]. Note that this second route uses some features of the Euclidean setting, such as the volume doubling property and Poincaré inequalities, whereas the first one does not; the latter can therefore be applied in a more general Riemannian setting without any further assumption, see Section 4 below.

We shall prove Proposition 2.2 by iterating the following.

Lemma 2.3 *Let $2 < p < p_1$, and assume* (1)*. Then there exists $\varepsilon > 0$ such that*

$$\|\nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1}\|_{p\to p} \le Ct^{-\varepsilon}\,\|\nabla(I+tH)^{-1/2}\|_{p\to p}$$

for all $t > 0$.

Proof. Set, for $t > 0$,

$$\begin{aligned} Z_t &= \nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1} \\ &= t\nabla(I+tH)^{-1}(H_0 - H)(I+tH_0)^{-1}, \end{aligned}$$

that is

$$Z_t = -t\nabla(I+tH)^{-1}\operatorname{div} a\nabla(I+tH_0)^{-1}. \tag{5}$$

Choose $p_2 \in (p, p_1)$ close enough to p so that

$$\varepsilon := \frac{D}{2}(p^{-1} - p_2^{-1}) < 1/2 \quad .$$

From the ultracontractivity estimate

$$\|e^{-tH_0}\|_{1\to\infty} \le Ct^{-D/2}, \ \forall\, t > 0, \tag{6}$$

we deduce by interpolation with the uniform boundedness of e^{-tH} on all L^r spaces, $1 \le r \le +\infty$, that

$$\|e^{-tH_0}\|_{p\to p_2} \le Ct^{-\varepsilon}. \tag{7}$$

Then observe that (1), interpolated with the similar L^2 estimate, which always holds, yields

$$\|\nabla e^{-tH_0}\|_{p_2\to p_2} \le Ct^{-1/2}.$$

Combining this with (7), we obtain

$$\|\nabla e^{-tH_0}\|_{p\to p_2} \le Ct^{-(1/2)-\varepsilon}.$$

Now recall that the multiplication by a is bounded from L^{p_2} to L^p as soon as $p^{-1} - p_2^{-1} \le q^{-1}$. We choose henceforth p_2 close enough to p so that this is the case. It follows that

$$\|a\nabla e^{-tH_0}\|_{p\to p} \le Ct^{-(1/2)-\varepsilon},\ \forall\, t > 0. \tag{8}$$

Now, since

$$(I + tH_0)^{-1} = \int_0^{+\infty} e^{-s}e^{-stH_0}\, ds,$$

one also has

$$\|a\nabla(I + tH_0)^{-1}\|_{p\to p} \le Ct^{-(1/2)-\varepsilon},\ \forall\, t > 0. \tag{9}$$

Using formula (5), we obtain the bound

$$\|Z_t\|_{p\to p} \le t\|\nabla(I + tH)^{-1/2}\|_{p\to p}\|(I + tH)^{-1/2}\operatorname{div}\|_{p\to p}\|a\nabla(I + tH_0)^{-1}\|_{p\to p}$$

hence, using (9),

$$\|Z_t\|_{p\to p} \le Ct^{(1/2)-\varepsilon}\|\nabla(I + tH)^{-1/2}\|_{p\to p}\|(I + tH)^{-1/2}\operatorname{div}\|_{p\to p}\ . \tag{10}$$

Let p' be the dual exponent to p, and recall that since $1 < p' < 2$ the Riesz transform $\nabla H^{-1/2}$ is bounded in $L^{p'}$ (see [5, chap.4]). Thus by duality we find that

$$\begin{aligned}\|(I + tH)^{-1/2}\operatorname{div}\|_{p\to p} &= \|\nabla(I + tH)^{-1/2}\|_{p'\to p'}\\ &\le t^{-1/2}\|\nabla(t^{-1} + H)^{-1/2}\|_{p'\to p'} \le Ct^{-1/2}\end{aligned}$$

for all $t > 0$; here we use the uniform boundedness of $H^{1/2}(t^{-1} + H)^{-1/2}$ on L^p, which follows from the analyticity of e^{-tH}. Substituting this estimate in (10) yields

$$\|Z_t\|_{p\to p} \le Ct^{-\varepsilon}\,\|\nabla(I + tH)^{-1/2}\|_{p\to p}$$

for $t > 0$, as desired. $\square$

Remark : Note that, because of (1) and the fact that $a \in L^\infty$, one always has

$$\|a\nabla e^{-tH_0}\|_{p\to p} \leq Ct^{-(1/2)}.$$

For $0 < t \leq 1$, this is a better estimate than (8). Thus, the ultracontractivity estimate (6) is not used for small time. This remark will be useful in Section 4 below.

Proof of Proposition 2.2 : Let us start with proving the first estimate in (3) for $0 < t \leq 1$; it follows from the analyticity of e^{-tH} on L^p that $(t + tH)^{1/2}(I + tH)^{-1}$ is uniformly bounded on L^p for $0 < t \leq 1$. Thus one may write, for $0 < t \leq 1$,

$$\|\nabla(I+tH)^{-1}\|_{p\to p} \leq \|\nabla(I+H)^{-1/2}\|_{p\to p}\|(I+H)^{1/2}(I+tH)^{-1}\|_{p\to p} \leq Ct^{-1/2},$$

as desired.

The proof of estimate (2) is based on the following claim: suppose that there is $\nu \in [0, 1/2)$ such that

$$\|\nabla(I + tH)^{-1}\|_{p\to p} = O(t^{-\nu})$$

for $t \geq 1$; then (with notation as in the proof of Lemma 2.3)

$$\|Z_t\|_{p\to p} = O(t^{-\varepsilon-\nu}) \quad , \quad \|\nabla(I + tH)^{-1}\|_{p\to p} = O(t^{-\nu'})$$

for $t \geq 1$, where ε is as in Lemma 2.3 and $\nu' := \min\{\nu + \varepsilon, 1/2\}$.

To prove the claim, we first show that

$$\|\nabla(\lambda + H)^{-1/2} - \nabla(I + H)^{-1/2}\|_{p\to p} \leq C\lambda^{\nu-(1/2)} \tag{11}$$

for all $\lambda \in (0, 1]$. Write

$$\begin{aligned}\nabla(\lambda + H)^{-1/2} - \nabla(I + H)^{-1/2} &= \frac{1}{\pi}\int_0^{+\infty} s^{-1/2}(e^{-\lambda s} - e^{-s})\,\nabla e^{-sH}\,ds \\ &= \frac{1}{\pi}\int_0^1 ... + \frac{1}{\pi}\int_1^{+\infty} ... \\ &= I_1 + I_2 \quad .\end{aligned}$$

Using the hypothesis of the claim, we have

$$\|\nabla e^{-sH}\|_{p\to p} \le \|\nabla(I+sH)^{-1}\|_{p\to p}\|(I+sH)e^{-sH}\|_{p\to p} \le Cs^{-\nu}$$

for all $s \ge 1$, the uniform boundedness of $(I+sH)e^{-sH}$ following again from the analyticity of e^{-sH} on L^p. Then

$$\begin{aligned}\|I_2\|_{p\to p} &\le C\int_1^{+\infty} s^{-1/2}e^{-\lambda s}s^{-\nu}\,ds \\ &\le C\int_0^{+\infty} s^{-(1/2)-\nu}e^{-\lambda s}\,ds \\ &= C'\lambda^{\nu-(1/2)}\end{aligned}$$

for all $\lambda \in (0,1]$; note that the last integral converges because $\nu < 1/2$. For $s \le 1$, the boundedness of $\nabla(I+H)^{-1/2}$ in L^p implies the estimate $\|\nabla e^{-sH}\|_{p\to p} \le Cs^{-1/2}$, by analyticity of e^{-sH} on L^p. Therefore, for all $\lambda \in (0,1]$,

$$\|I_1\|_{p\to p} \le C\int_0^1 s^{-1}\left(e^{-\lambda s} - e^{-s}\right)\,ds \le C'$$

using $0 \le e^{-\lambda s} - e^{-s} \le Cs$ for all $\lambda, s \in (0,1]$. This proves (11).

From the assumed boundedness of $\nabla(I+H)^{-1/2}$, and the fact that $\nu < 1/2$, it follows that

$$\|\nabla(\lambda+H)^{-1/2}\|_{p\to p} \le C\lambda^{\nu-(1/2)},$$

for all $\lambda \in (0,1]$, hence, setting $\lambda = t^{-1}$,

$$\|\nabla(I+tH)^{-1/2}\|_{p\to p} \le Ct^{-\nu}$$

for all $t \ge 1$. Lemma 2.3 then yields

$$\|\nabla(I+tH)^{-1} - \nabla(I+tH_0)^{-1}\|_{p\to p} \le Ct^{-\varepsilon-\nu} \tag{12}$$

for $t \ge 1$, that is, the first part of the claim. Notice that

$$\|\nabla(I+tH_0)^{-1}\|_{p\to p} \le Ct^{-1/2},$$

which follows from $\|\nabla e^{-tH_0}\|_{p\to p} \le Ct^{-1/2}$ as in the proof of (9). Together with (12), this yields $\|\nabla(I+tH)^{-1}\|_{p\to p} \le Ct^{-\nu'}$ for $t \ge 1$, which completes the proof of the claim.

Now observe that the hypothesis of the claim is satisfied with $\nu = 0$, as a consequence of the boundedness of $\nabla(I+H)^{-1/2}$. Thus we may repeatedly apply the claim, increasing the value of ν with each iteration. The exponent ν' will eventually equal $1/2$, which proves the first assertion in (3) for the remaining case $t \geq 1$; then taking ν arbitrarily close to $1/2$ in the next step yields

$$\|Z_t\|_{p\to p} = O(t^{-\delta-(1/2)})$$

for any $\delta \in (0, \varepsilon)$, which proves estimate (2).

The second assertion in (3) follows from the first one by using the uniform boundedness of $(I + tH)e^{-tH}$ on L^p. Proposition 2.2 is proved. □

Remark : The Berry-Esseen estimate (2), for gradients of the resolvents, is one of our central results. It is interesting to observe that Berry-Esseen estimates without gradients also hold in our situation, and are easier to obtain.

In fact, if $p_1 \in (2, +\infty]$ is such that (1) holds, then choosing $s \in (2, p_1)$ close to p_1, an argument similar to that of Lemma 2.3 yields

$$\begin{aligned} & \|(I+tH)^{-1} - (I+tH_0)^{-1}\|_{p_1\to p_1} \\ \leq\ & t\|(I+tH)^{-1} \operatorname{div}\|_{s\to p_1}\|a\|_{p_1\to s}\|\nabla(I+tH_0)^{-1}\|_{p_1\to p_1} \\ \leq\ & Ct^{-\delta} \end{aligned}$$

for some $\delta > 0$ and all $t \geq 1$. One can interpolate this with the obvious estimate $\|(I+tH)^{-1} - (I+tH_0)^{-1}\|_{r\to r} \leq 2$, for $r \in [1, +\infty]$, to see that for each $p \in (1, +\infty)$ there exists $\delta_p > 0$ such that the Berry-Esseen estimate

$$\|(I+tH)^{-1} - (I+tH_0)^{-1}\|_{p\to p} \leq Ct^{-\delta_p}$$

holds for all $t \geq 1$. In the case where (1) holds with $p_1 = +\infty$, the last estimate holds for $p = +\infty$ and also, by duality, for $p = 1$.

One can easily vary these ideas to obtain Berry-Esseen estimates from L^p to L^r for suitable p, r, or Berry-Esseen estimates for the semigroup difference $e^{-tH} - e^{-tH_0}$. Moreover, analogous estimates hold in the more general settings of Sections 3 and 4 below. We shall leave the details to the reader.

3 Weighted divergence form operators

In this section we show how to extend the results of the preceding section to weighted operators in $\mathbb{R}^D$ of the form

$$H = -\sigma^{-1} \operatorname{div} \sigma A \nabla, \quad H_0 = -\sigma_0^{-1} \operatorname{div} \sigma_0 A_0 \nabla.$$

Here A, A_0 are as before, and $\sigma, \sigma_0 \colon \mathbb{R}^D \to (0, +\infty)$ are positive, measurable weight functions satisfying uniform upper and lower bounds

$$C^{-1} \leq \sigma(x), \sigma_0(x) \leq C \tag{13}$$

for all $x \in \mathbb{R}^D$. The operators H, H_0, again defined through their quadratic forms, for instance

$$(Hf, f)_{L^2(\sigma dx)} = \int_{\mathbb{R}^D} A(x) \nabla f(x) \cdot \nabla f(x) \sigma(x)\, dx,$$

are self-adjoint respectively in the weighted spaces $L^2(\sigma dx)$ and $L^2(\sigma_0 dx)$ where dx denotes Lebesgue measure. We could also express

$$H = -\operatorname{div}_\sigma A \nabla, \quad H_0 = -\operatorname{div}_{\sigma_0} A_0 \nabla$$

where the operator $\operatorname{div}_\sigma := \sigma^{-1} \circ \operatorname{div} \circ \sigma$ is formally adjoint to $-\nabla$ with respect to the measure σdx (and with $\operatorname{div}_{\sigma_0}$ defined similarly).

It is important for us to observe that, because of (13), one has for all p the equality of vector spaces

$$L^p(\sigma dx) = L^p(\sigma_0 dx) = L^p(dx)$$

with the different norms being equivalent. Thus in this section, there will be no harm in writing simply L^p for any of these spaces, with similar simplifications for operator norms.

Theorem 3.1 *Let H, H_0 be as above and assume that $A - A_0 \in L^q$ and $\sigma - \sigma_0 \in L^q$ for some $q \in [1, +\infty)$. Then the statements of Theorems* 2.1 *and Proposition* 2.2 *remain valid.*

In the proof of this result, the only substantial difference from the unweighted case occurs in the proof of Lemma 2.3, which we generalize as follows.

Lemma 3.2 *Let $2 < p < p_1$ and assume* (1)*. Then for some $\varepsilon > 0$ and all $t \geq 1$, the estimate of Lemma* 2.3 *holds for the weighted operators H and H_0.*

Proof. We can express

$$\begin{aligned} H - H_0 &= \sigma^{-1}\operatorname{div}(\sigma_0 A_0 - \sigma A)\nabla + (\sigma_0^{-1} - \sigma^{-1})\operatorname{div}\sigma_0 A_0 \nabla \\ &= \operatorname{div}_\sigma(\sigma_0\sigma^{-1}A_0 - A)\nabla - (1 - \sigma_0\sigma^{-1})H_0. \end{aligned}$$

Hence we may write

$$\begin{aligned} Z_t &:= \nabla(I + tH)^{-1} - \nabla(I + tH_0)^{-1} \\ &= t\nabla(I + tH)^{-1}(H_0 - H)(I + tH_0)^{-1} \\ &= -t\nabla(I + tH)^{-1}\operatorname{div}_\sigma(\sigma_0\sigma^{-1}A_0 - A)\nabla(I + tH_0)^{-1} \\ &\quad +t\nabla(I + tH)^{-1}(1 - \sigma_0\sigma^{-1})H_0(I + tH_0)^{-1} \\ &= J_1 + J_2. \end{aligned}$$

Just as in the proof of Lemma 2.3, we can estimate J_1 by

$$\|J_1\|_{p\to p} \leq Ct^{-\varepsilon_1}\|\nabla(I + tH)^{-1/2}\|_{p\to p}$$

for some $\varepsilon_1 > 0$ and all $t > 0$, by using an estimate of type (9), the fact that

$$\sigma_0\sigma^{-1}A_0 - A = (\sigma_0 - \sigma)\sigma^{-1}A_0 + (A_0 - A) \in L^q \cap L^\infty,$$

and by noting that $\|(I + tH)^{-1/2}\operatorname{div}_\sigma\|_{p\to p} \leq Ct^{-1/2}$ (which is, as before, a consequence of duality and the fact that $\nabla H^{-1/2}$ is bounded in $L^{p'}$).

The term J_2 is of a new type, and can be estimated as follows. Choose $p_3 \in (2, p)$ close enough to p so that multiplication by $1 - \sigma_0\sigma^{-1} \in L^q \cap L^\infty$ maps L^p into L^{p_3}, and so that $\varepsilon_2 := (D/2)(p_3^{-1} - p^{-1}) < 1/2$. We have

$$\|e^{-tH}(1 - \sigma_0\sigma^{-1})\|_{p\to p} \leq C\|e^{-tH}\|_{p_3\to p} \leq C't^{-\varepsilon_2},$$

and the formula

$$(I+tH)^{-1/2} = \frac{1}{\pi}\int_0^{+\infty} s^{-1/2}e^{-s}e^{-stH}\,ds$$

then shows that

$$\|(I+tH)^{-1/2}(1-\sigma_0\sigma^{-1})\|_{p\to p} \le C't^{-\varepsilon_2},\ \forall\, t>0. \tag{14}$$

Therefore

$$\begin{aligned}\|J_2\|_{p\to p} &\le t\|\nabla(I+tH)^{-1/2}\|_{p\to p}\|(I+tH)^{-1/2}(1-\sigma_0\sigma^{-1})\|_{p\to p}\|H_0(I+tH_0)^{-1}\|_{p\to p}\\ &\le Ct\|\nabla(I+tH)^{-1/2}\|_{p\to p}t^{-\varepsilon_2}t^{-1}\end{aligned}$$

for $t>0$, where we used the elementary bound $\|H_0(I+tH_0)^{-1}\|_{p\to p} \le Ct^{-1}$. Thus Lemma 3.2 follows for $t\ge 1$ with $\varepsilon := \min\{\varepsilon_1,\varepsilon_2\}$. □

4 Riemannian manifolds

Let M be a non-compact, connected smooth manifold, and let $G_0 = \langle,\rangle_0$ and $G = \langle,\rangle$ be two complete Riemannian metrics on M. The respective Riemannian gradients, divergences and measures on M for G_0 and G will be denoted by ∇_0, ∇, div_0, div, $d\mu_0$, $d\mu$, and the corresponding (positive) Laplace operators are

$$H_0 = -\operatorname{div}_0\nabla_0,\quad H = -\operatorname{div}\nabla.$$

We shall assume that the metrics G and G_0 are quasi-isometric in the sense that the associated lengths $|\cdot|$, $|\cdot|_0$ on tangent vectors satisfy

$$C^{-1}|v| \le |v|_0 \le C|v| \tag{15}$$

for all $x\in M$ and $v\in T_xM$. This implies (see Proposition 4.3 below) that for each p one has

$$L^p(d\mu_0) = L^p(d\mu)$$

as vector spaces, with equivalent norms; in this section we will sometimes simply write L^p for these spaces. Moreover (see again Proposition 4.3) the lengths $|\nabla f|$ and $|\nabla_0 f|_0$ are equivalent pointwise, so that, for example, boundedness of $\nabla H^{-1/2}$ in L^p is equivalent to boundedness of $\nabla_0 H^{-1/2}$ in L^p. Remarks such as these will be used freely.

Our second main assumption is that G is an L^q perturbation of G_0 in the following sense. Define an automorphism $A = (A(x))_{x\in M}$ of the tangent bundle TM by requiring that

$$\langle A(x)v, w\rangle = \langle v, w\rangle_0 \tag{16}$$

for all $x \in M$ and $v, w \in T_x M$. We assume that

$$\|A - I\|_{L^q(d\mu_0, G_0)} := \left(\int_M |A(x) - I|_0^q \, d\mu_0(x)\right)^{1/q} < \infty \tag{17}$$

for some $q \in [1, +\infty)$, where $|\cdot|_0$ denotes, say, the Hilbert-Schmidt norm on $\mathcal{L}(T_x M)$ induced by G_0 (of course, one could equivalently replace the G_0 objects by G objects in assumption (17)).

We shall also suppose that for some constant $D > 0$ one has

$$\|e^{-tH_0}\|_{1\to\infty} \le Ct^{-D/2}, \ t \ge 1. \tag{18}$$

It is then a consequence of standard results that thanks to (15), the semigroup e^{-tH} also satisfies (18).

Conditions under which assumption (18) is satisfied can be found in [13, Theorem 1.1]. For example, if one has the volume doubling property together with scaled Poincaré inequalities, both for small geodesic balls, as well as non-collapsing of the volume of balls of radius one, namely

$$\inf_{x\in M} V(x, 1) > 0,$$

then one has (18) with $D = 1$. Ricci curvature bounded from below by a negative constant together with positive injectivity radius of course ensures this. A slow decay of the volume of balls of radius 1 as x goes to infinity in M may even be allowed, at the expense of a smaller exponent $D > 0$.

Theorem 4.1 *Let $p_0 > 2$ and p_0' the conjugate exponent. Adopt assumptions* (15), (17) *and* (18)*. Assume that the global Riesz transform $\nabla_0 H_0^{-1/2}$*

and the local Riesz transform $\nabla(I+H)^{-1/2}$ *are bounded on* L^p *for all* $p \in (2, p_0)$*, and that the global Riesz transform* $\nabla H^{-1/2}$ *is bounded in* $L^{p'}$ *for all* $p' \in (p_0', 2)$*. Then* $\nabla H^{-1/2}$ *is bounded in* L^p *for all* $p \in (2, p_0)$*.*

Sufficient conditions for the local Riesz transform to be bounded on L^p for $p > 2$ can be found in [3, Theorem 1.7] ; Ricci curvature bounded from below by a negative constant is enough, as was shown earlier in [6].

Sufficient conditions for the global Riesz transform to be bounded on L^p for $1 < p < 2$ can be found in [8, Theorem 1.1]. They involve the volume doubling property and a natural upper bound for the heat kernel.

Example : Let M be an n-dimensional complete Riemannian manifold, with $n > 2$, endowed with two metrics G and G_0 satisfying assumptions (15) and (17). Assume that the Ricci curvature for G is bounded from below by a negative constant, and that M satisfies

$$V(x,r) \leq Cr^n, \ \forall\, x \in M, \ r > 0,$$

and

$$\|f\|_{\frac{2n}{n-2}} \leq C\|\nabla f\|_2, \ \forall\, f \in C_0^\infty(M).$$

With the notation above, if for some $p_0 > 2$ the global Riesz transform $\nabla_0 H_0^{-1/2}$ is bounded on L^p for all $p \in (2, p_0)$, then $\nabla H^{-1/2}$ is bounded in L^p for all $p \in (2, p_0)$.

As in Section 2, Theorem 4.1 follows from a Berry-Esseen type estimate.

Proposition 4.2 *Let* $2 < p < p_1$ *and suppose that*

$$\|\nabla_0 e^{-tH_0}\|_{p_1 \to p_1} \leq Ct^{-1/2}$$

for all $t > 0$*. Assume that* $\nabla(I+H)^{-1/2}$ *is bounded in* L^p*, and that* $\nabla H^{-1/2}$ *is bounded in* $L^{p'}$ *where* p' *is dual to* p*. Then there exists an* $\varepsilon > 0$ *such that*

$$\|\nabla_0(I+tH)^{-1} - \nabla_0(I+tH_0)^{-1}\|_{p \to p} \leq Ct^{-\varepsilon-1/2}$$

for all $t \geq 1$*. In addition,*

$$\|\nabla(I+tH)^{-1}\|_{p\to p} \leq Ct^{-1/2} \quad \text{and } \|\nabla e^{-tH}\|_{p \to p} \leq Ct^{-1/2}$$

for all $t > 0$*.*

The proofs of Theorem 4.1 and Proposition 4.2 follow closely the arguments of previous sections. We need the following technical proposition expressing various relations between the G_0 and G objects on M; note that similar formulae relating two Riemannian metrics were given by Barbatis in [7]. We give a proof of the proposition at the end of the section.

Proposition 4.3 *Assume* (15) *and* (17). *With A defined by* (16) *and with $\sigma := (\det A)^{-1/2}\colon M \to (0,+\infty)$, one has the following identities:*

$$\nabla f = A\nabla_0 f \tag{19}$$

$$d\mu = \sigma d\mu_0 \tag{20}$$

$$\operatorname{div} X = \sigma^{-1}\operatorname{div}_0(\sigma X) \tag{21}$$

$$Hf = -\sigma^{-1}\operatorname{div}_0(\sigma A\nabla_0 f) \tag{22}$$

for all $f \in C^\infty(M)$ and smooth vector fields X on M.

Moreover, $A(x)$ is a positive self-adjoint operator in T_xM with respect to either G or G_0, and for some $C \geq 1$,

$$C^{-1} \leq A(x) \leq C$$

for all $x \in M$, in the sense of either G or G_0. Then for some $C \geq 1$ one has

$$C^{-1} \leq \sigma(x) \leq C$$

$$C^{-1}|(\nabla f)(x)| \leq |(\nabla_0 f)(x)|_0 \leq C|(\nabla f)(x)|$$

for all $x \in M$ and $f \in C^\infty(M)$. One also has $L^p(d\mu) = L^p(d\mu_0)$ as vector spaces, with equivalent norms.

Finally, $1 - \sigma^\delta \in L^q$, for all $\delta \in \mathbb{R}$.

The following result is the analogue, for the Riemannian setting, of Lemmas 2.3 and 3.2.

Lemma 4.4 *Let $2 < p < p_1$ and assume that $\|\nabla_0 e^{-tH_0}\|_{p_1\to p_1} \leq Ct^{-1/2}$ for all $t > 0$. Suppose that $\nabla H^{-1/2}$ is bounded in $L^{p'}$. Then there exists $\varepsilon > 0$ such that*

$$\|\nabla_0(I+tH)^{-1} - \nabla_0(I+tH_0)^{-1}\|_{p\to p} \leq Ct^{-\varepsilon}\|\nabla_0(I+tH)^{-1/2}\|_{p\to p}$$

for all $t \geq 1$.

Let us sketch the proof of Lemma 4.4. From the expressions of Proposition 4.3 one computes, similarly to the proof of Lemma 3.2,

$$\begin{aligned} H - H_0 &= \sigma^{-1} \operatorname{div}_0 (I - \sigma A)\nabla_0 + (1 - \sigma^{-1}) \operatorname{div}_0 \nabla_0 \\ &= \operatorname{div}(\sigma^{-1} I - A)\nabla_0 - (1 - \sigma^{-1}) H_0. \end{aligned}$$

We can then write

$$\begin{aligned} Z_t &:= \nabla_0 (I + tH)^{-1} - \nabla_0 (I + tH_0)^{-1} \\ &= -t\nabla_0 (I + tH)^{-1} \operatorname{div}(\sigma^{-1} I - A)\nabla_0 (I + tH_0)^{-1} \\ &\quad + t\nabla_0 (I + tH)^{-1} (1 - \sigma^{-1}) H_0 (I + tH_0)^{-1} \\ &= J_1 + J_2. \end{aligned}$$

The terms J_1 and J_2 can be estimated by the same techniques as in the proof of Lemma 3.2. For this, one uses that

$$\sigma^{-1} I - A = \sigma^{-1}(1 - \sigma) I + (I - A) \in L^q \cap L^\infty$$

and $\sigma^{-1} - 1 \in L^q \cap L^\infty$, as well as the analogues of (9) and (14)

$$\|(\sigma^{-1} I - A)\nabla_0 (I + tH_0)^{-1}\|_{p \to p} \leq C t^{-(1/2) - \varepsilon_1} \tag{23}$$

and

$$\|(I + tH)^{-1/2} (1 - \sigma^{-1})\|_{p \to p} \leq C t^{-\varepsilon_2} \tag{24}$$

for $t > 0$ and some $\varepsilon_1, \varepsilon_2 > 0$. Note that in the proofs of (23) and (24), no ultracontractivity estimate is needed for small t ; the uniform L^p boundedness on the semigroup under consideration or the assumption on its gradient are sufficient. See the remark at the end of the proof of Lemma 2.3. Therefore (18) and the analogue of (18) for e^{-tH} suffice to derive estimates (23) and (24). We omit further details. □

Given Lemma 4.4, the proofs of Theorem 4.1 and Proposition 4.2 follow closely the arguments of Section 2, and so we leave further details to the reader.

The potential applications of our perturbation result are more obvious in the range $p_0 > 2$; we can however record that a statement similar to that of Theorem 4.1 also holds for $p_0 \in (1,2)$. The proofs are the same, up to obvious modifications in the choice of exponents.

Proof of Proposition 4.3. While some results of this Proposition could be extracted from [7], we give direct proofs for the reader's convenience.

The metrics G, G_0 respectively determine bundle isomorphisms $\alpha\colon TM \to T^*M$, $\alpha_0\colon TM \to T^*M$, by setting

$$(\alpha(x)v)(w) = \langle v, w\rangle, \quad (\alpha_0(x)v)(w) = \langle v, w\rangle_0$$

for all $x \in M$, $v, w \in T_xM$. Equation (16) rewrites as $(\alpha(x)A(x)v)(w) = (\alpha_0(x)v)(w)$, which shows that

$$A = \alpha^{-1}\alpha_0.$$

Since the gradients are given by $\nabla f = \alpha^{-1}(df)$ and $\nabla_0 f = \alpha_0^{-1}(df)$, then (19) follows.

Next, in local coordinates G and G_0 are represented by matrices g, g_0 respectively, and it is well known that

$$d\mu = (\det g)^{1/2}dx, \quad d\mu_0 = (\det g_0)^{1/2}dx$$

where dx denotes Lebesgue measure in the local coordinates. But one easily checks that A has matrix $g^{-1}g_0$ in local coordinates, so that $d\mu = (\det A)^{-1/2}d\mu_0$ and (20) is proved.

For (21), we compute for any $f \in C_0^\infty(M)$ that

$$\begin{aligned}
-(\operatorname{div} X, f)_{L^2(d\mu)} &= \int_M \langle X, \nabla f\rangle d\mu \\
&= \int_M \langle \sigma X, A\nabla_0 f\rangle d\mu_0 \\
&= \int_M \langle \sigma X, \nabla_0 f\rangle_0 d\mu_0 \\
&= -(\operatorname{div}_0(\sigma X), f)_{L^2(d\mu_0)} \\
&= -(\sigma^{-1}\operatorname{div}_0(\sigma X), f)_{L^2(d\mu)}
\end{aligned}$$

which implies (21). From (19), (21) and the equation $H = -\operatorname{div}\nabla$ we obtain (22).

The remaining assertions for $A(x)$, and the estimates on $\sigma(x)$ and the gradients, can easily be obtained from the definitions and from (15).

Finally, one has $\det A - 1 \in L^q$ as a consequence of $A - I \in L^q$ and the general inequality

$$|\det B - 1| \leq C' \|B - I\|$$

for linear operators B in an N-dimensional inner product space, where C' is a constant depending only on N and on the Hilbert-Schmidt norm $\|B\|$. Since $C^{-1} \leq \det A \leq C$ for some $C > 1$, it is then elementary that $(\det A)^\delta - I \in L^q$ for any $\delta \in \mathbb{R}$. □

Remark : As pointed out by the referee, if one follows the constants, the proof of Theorem 4.1 yields an estimate of $\|\nabla H^{-1/2} - \nabla H_0^{-1/2}\|_{p\to p}$ in terms of $\|A - I\|_q$.

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