

# A SINGULAR CRITICAL POTENTIAL FOR THE SCHRÖDINGER OPERATOR

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## Abstract

Consider a real potential  $V$  on  $\mathbf{R}^d$ ,  $d \geq 2$ , and the Schrödinger equation:

$$(LS) \quad i\partial_t u + \Delta u - Vu = 0, \quad u|_{t=0} = u_0 \in L^2.$$

In this paper, we investigate the minimal local regularity of  $V$  needed to get local in time dispersive estimates (such as local in time Strichartz estimates or local smoothing effect with gain of  $1/2$  derivative) on solutions of (LS). Preceding works show some dispersive properties when  $V$  (small at infinity) is in  $L^{d/2}$ , or in spaces just a little larger but with a smallness condition on  $V$  (or at least on its negative part).

In this work, we prove the critical character of this results by constructing a positive potential  $V$  which has compact support, bounded outside 0 and of the order  $(\log|x|)^2/|x|^2$  near 0. The lack of dispersiveness comes from the existence of a sequence of quasimodes for the operator  $P := -\Delta + V$ .

The elementary construction of  $V$  consists in sticking together concentrated, truncated potential wells near 0. This yields a potential oscillating with infinite speed and amplitude at 0, such that the operator  $P$  admits a sequence of quasi-modes of polynomial order whose support concentrates on the pole.

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## 1. INTRODUCTION

Consider a Schrödinger operator:

$$(1) \quad P := -\Delta + V, \quad \Delta := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2},$$

on  $\mathbf{R}^d$ ,  $d \geq 1$ , where  $V$  is a time-independent, real potential, small at infinity, with a small negative part, and the initial value problem for the Schrödinger equation:

$$(2) \quad \begin{cases} i\partial_t U - PU = 0 \\ U|_{t=0} = U_0 \in L^2(\mathbf{R}^d). \end{cases}$$

For potentials  $V$  which are not too singular, the solutions of equation (2) enjoy many dispersive properties, such as the classical dispersion estimate, Strichartz estimates and the local smoothing effect (introduced by Constantin-Saut [7], Sjölin [14], Vega [16]):

$$(3) \quad \|\chi U\|_{L^2([0,T[,H^{1/2}(\mathbf{R}^d))} \leq C\|U_0\|_{L^2(\mathbf{R}^d)}, \quad \chi \in C_0^\infty(\mathbf{R}^d),$$

where  $T$  may be finite, and in some cases infinite, and  $C$  may depend on  $V$ ,  $\chi$  and  $T$ .

These well-known properties are essential in the description of the solutions of (2) and the study of related non-linear equations. It is natural to look for the minimal assumptions to be made on  $V$  for the dispersive properties to hold. This issue has two aspects, the boundedness of  $V$  at infinity and its local regularity. In this paper, we shall ignore the first question, taking only potentials that decay fast enough for large  $|x|$ , and only investigate the second one. One of our motivation is that the linearization of non-linear Schrödinger equations, near a singular stationary solution may yield very harsh potentials. We now recall some important results on the subject.

First note that for potentials

$$V \in L^{d/2}, \quad d \geq 3 \text{ and } V \in L^p, \quad p > 1, \quad d = 2$$

small at infinity, it is easy to deduce local in time Strichartz estimates from the free equation estimates using Duhamel's formula.

In [13], Ruiz and Vega, using perturbative arguments, showed that for this potentials, in dimension  $d \geq 3$  the smoothing effect (3) also holds. Furthermore, there results handle more singular potentials belonging to some Morrey-Campanato spaces, which are (from a local point of view), strictly larger than the Lorentz space  $L_w^{d/2}$  (and thus than  $L^{d/2}$ ), but strictly contained in  $\cap_{p < d/2} L^p$ . In this case a smallness condition on the potential (or at least on its most singular part) is needed. The simpler example of this kind is the inverse square potential:

$$V_a := \frac{a}{|x|^2}, \quad |a| < \varepsilon_d,$$

where  $\varepsilon_d$  is a constant depending on  $d$ .

In [5], Burq, Planchon, Stalker and Tahvildar-Zadeh showed Strichartz estimates and smoothing effect on the Schrödinger equation with the potential  $V_a$  (see also [9] for the multipolar case), assuming only the following positivity property on  $a$ :

$$a + (d/2 - 1)^2 > 0.$$

When  $a + (d/2 - 1)^2 < 0$ , there is a sequence of eigenvalues going to  $-\infty$ , such that the sequence of the corresponding stationary solutions invalidate the dispersive-type estimates. Therefore,

these critical potentials may not be considered as perturbations of the Laplacien, but rather as operators of the same order, which may change radically the behaviour of  $P$ .

In a recent article [12], Rodniansky and Schlag have proven dispersive-type decay of solutions of Schrödinger equations, in dimension 3, with potentials that are in a Kato class which is, again, locally strictly contained in  $\cap_{p < d/2} L^p$ , but larger than  $L^{d/2}$ . In their article, the authors make a smallness assumption on the negative, singular part of the potential, which is similar to the positivity assumption on  $a$ .

These results suggest that the critical class of potentials from the point of view of local regularity is a little larger than  $L^{d/2}$ , but to our knowledge little is known about the optimality of the assumptions on the positive part of  $V$ . Indeed nothing in the above works suggest that a positive, very singular potential may prevent solutions of (2) to enjoy dispersive properties.

In this paper, we answer to this question, constructing a unipolar, positive potential, with compact support, whose pole is of the order  $\frac{(\log|x|)^2}{|x|^2}$  near 0, and such that local in time dispersive estimates do not hold. This construction shows the quasi-optimality of the works mentioned above ([13], [5], [9], [12], see also the paper of Goldberg [10]). The lack of dispersion comes from the existence of a sequence of quasi-modes for the operator  $P$  (i.e approximate solutions of the equation  $Pu - \lambda u = 0$  with  $\lambda$  going to infinity). Note that the existence of a finite number of eigenfunctions would show that global in time dispersive estimates cannot hold. Our result is stronger as it shows also the impossibility of **local in time** estimates (which are really high-frequency types estimates). Recall also that in general, there is no uniformity of the estimates with respect to the norm of the potential in the critical spaces. Thus it is important to construct a single  $V$  invalidating the dispersive properties, rather than giving an example of a sequence of potentials (even with an uniform bound).

**Theorem 1** (Existence of quasi-modes). *Let  $d \geq 1$ ,  $N \geq 0$  be integers. There exist:*

- a radial, positive potential  $V$  on  $\mathbf{R}^d$ , which has compact support and such that:

$$(4) \quad V \in C^\infty(\mathbf{R}^d \setminus \{0\})$$

$$(5) \quad \frac{|\log r|^2}{Cr^2} \leq V(r) \leq \frac{C|\log r|^2}{r^2}, \quad r \leq r_0,$$

- an increasing sequence  $(\lambda_n)_{n \geq n_0}$  of positive real numbers, diverging to  $+\infty$ ,
- a sequence of radial  $C^\infty$  functions  $(u_n)_{n \geq n_0}$ , whose support is of the following form:

$$\left\{ c_1 \frac{\log \lambda_n}{\lambda_n} \leq r \leq c_2 \frac{\log \lambda_n}{\lambda_n} \right\}, \quad 0 < c_1 < c_2;$$

such that:

$$(6) \quad (-\Delta + V)u_n - \lambda_n^2 u_n = f_n$$

$$(7) \quad \|u_n\|_{L^1} = 1$$

$$(8) \quad \forall j \in \mathbb{N}, \quad \left\| \frac{d^j}{dr^j} f_n \right\|_{L^\infty} = O(\lambda_n^{j-N}), \quad n \rightarrow +\infty.$$

**Corollary 1.** *Let  $N \geq 2$ ,  $P = -\Delta + V$ , where  $V$  is the potential of the preceding theorem, and  $\chi$  a function in  $C_0^\infty(\mathbf{R}^d)$  which does not vanish in 0. Then the solutions of (2) do not enjoy any of the classical dispersive properties:*

**lack of local Strichartz estimates:**

$$(9) \quad \forall q, q_0 \in [1, +\infty], q > q_0, \forall T > 0, \forall K > 0, \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \\ \|\chi U(t)\|_{L^1([0, T], L^q(\mathbf{R}^d))} > K \|U_0\|_{L^{q_0}(\mathbf{R}^d)};$$

**lack of local smoothing effect:**

$$(10) \quad \forall \sigma > 0 \forall T > 0, \forall K > 0, \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \|\chi U(t)\|_{L^1([0, T], H^\sigma(\mathbf{R}^d))} > K \|U_0\|_{L^2(\mathbf{R}^d)};$$

**lack of local dispersion:**

$$(11) \quad \forall q \in [1, +\infty], \forall T > 0, \forall K > 0, \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \|\chi U(T)\|_{L^q(\mathbf{R}^d)} > K \|U_0\|_{L^{q'}(\mathbf{R}^d)};$$

**lack of Strichartz estimates with loss of derivative:**

$$(12) \quad \forall q \in [1, +\infty], \forall \sigma \in [0, 1], \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d}, \\ \forall T > 0, \forall K > 0, \exists U_0 \in C_0^\infty(\mathbf{R}^d \setminus \{0\}), \|\chi U(t)\|_{L^1([0, T], L^q(\mathbf{R}^d))} > K \|U_0\|_{D(P^{\sigma/2})}.$$

In the preceding statements,  $U(t)$  is the solution of the equation (2) with initial value  $U_0$ , and  $q'$  the conjugate exponent of  $q$  which is defined by:  $1/q + 1/q' = 1$ .

**Remarks.** • If  $d > 2$ , the hypothesis (4) and (5) imply:

$$V \in \bigcap_{p < d/2} L^p.$$

- The sequence  $(f_n)_{n \geq n_0}$  invalidates the classical non-trapping type, resolvent estimates (cf [2]), which imply the local smoothing effect (3). Indeed one may see  $V$  as a trapping potential, as it concentrates, on the pole 0, the energy of a sequence of quasi-modes. In similar settings, the equivalence between a non-trapping type assumption and the local smoothing effect is by no well understood (see for example [3]). As will be clear in the proof, this trapping property is a consequence of the very fast oscillations of  $V$  near 0.
- It will be clear in the proof of the theorem than one may construct quasi-modes of infinite order (i.e. such that (8) holds with  $O(\lambda_n^{-\infty})$  instead of  $O(\lambda_n^{j-N})$ ) by taking a singularity just a bit stronger:

$$\frac{|\log r|^{2+\varepsilon}}{Cr^2} \leq V(r) \leq \frac{C|\log r|^{2+\varepsilon}}{r^2}, \quad \varepsilon > 0.$$

With a still stronger singularity, one may force  $f_n$  to be exponentially decreasing in  $-\lambda_n$ .

- Classical Strichartz estimates:

$$\|U\|_{L^p([0, T], L^q(\mathbf{R}^d))} \leq C \|U_0\|_{L^2(\mathbf{R}^d)}, \quad p > 2, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

are invalidated by (9), with the exception of the case  $q = 2, p = +\infty$  which is always true by the  $L^2$  conservation law. Likewise, (12) shows that Strichartz estimates with loss of derivative (see [4]) of the form:

$$\|U\|_{L^p([0, T], L^q(\mathbf{R}^d))} \leq C \|U_0\|_{D(P^{\sigma/2})}, \quad \frac{1}{2} - \frac{1}{q} > \frac{\sigma}{d}, \quad \sigma \in [0, 1],$$

are false. In this inequality, the limit case  $\sigma/d = 1/2 - 1/q$  is an immediate consequence of the Sobolev inclusion

$$L^q(\mathbf{R}^d) \subset H^\sigma(\mathbf{R}^d) \subset D(P^{\sigma/2}),$$

and of the conservation of the norm  $D(P^{\sigma/2})$  of any solution of (2). Notice that this last inclusion implies that (12) is still true in usual Sobolev spaces, i.e when  $D(P^{\sigma/2})$  is replaced by  $H^{\sigma}(\mathbf{R}^n)$ .

- The potential  $V$  and quasi-modes  $u_n$  being of compact support, as close to 0 as desired, the preceding counter-example is still valid in any regular domain, for a Laplace operator with Dirichlet or Neumann boundary conditions.
- It is also possible to deduce from theorem 1 negative results on the solutions of the wave equation (see the remark in the end of the proof of the corollary):

$$(13) \quad \partial_t^2 u + Pu = 0.$$

For example, the usual Strichartz and dispersive estimates, and the uniform decay of local energy do not hold. For positive results in this direction, see [13], or the recent work of D'Ancona and Pierfelice [8], in dimension 3. Their assumption on  $V$ , taken in a critical Kato class, is very close to the one of Rodniansky and Schlag in [12].

- Examples of linear Schrödinger equations which do not admit the classical dispersion inequality or Strichartz estimates are given in [1] and [6]. In these two cases, the particular behavior of the equation arises from the metric which defines the Laplace operator. Our construction is close to the one of C. Castro and E. Zuazua in [6], where the authors investigate the critical regularity of the metric defining the Laplace operator in order to get certain control properties on the wave and Schrödinger equations. Let us mention that the idea to use quasi-modes in relation with Strichartz inequalities is due to M. Zworski (see [15]).

Finally, we would like to point out that for the potential  $V$  introduced in the theorem, one may define the operator  $P$  without ambiguity. The operator  $-\Delta + V$  has a natural meaning on  $C_0^\infty(\mathbf{R}^d \setminus \{0\})$ , and is essentially self-adjoint on this space, under a positivity assumption implied by (5). We call  $P$  its unique self-adjoint extension, which is of course positive. We refer to Reed and Simon [11, Th X.30] for any precision.

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Section 2 of the paper is devoted to the proof of the corollary, and section 3 to that of the theorem.

## 2. PROOF OF THE COROLLARY.

According to Hölder's inequality, for any function  $v$  on  $\mathbf{R}^d$ , with a finite volume support:

$$(14) \quad \|v\|_{L^{q_0}} \leq B^{1/q_0 - 1/q} \|v\|_{L^q}, \quad q > q_0,$$

where  $B$  is the volume of the support. Let  $U_n(t) = e^{-i\lambda_n^2 t} u_n$ . Then:

$$(15) \quad \begin{aligned} i\partial_t U_n - P U_n &= -e^{-i\lambda_n^2 t} f_n, \quad U_n|_{t=0} = u_n \\ U_n(t) &= e^{-itP} u_n + i \int_0^t e^{-i\lambda_n^2 s} e^{i(s-t)P} f_n ds. \end{aligned}$$

Assume that (9) is false. According to (15):

$$\int_0^T \|\chi U_n(t)\|_{L^q(\mathbf{R}^d)} dt \leq \int_0^T \|\chi e^{-itP} u_n\|_{L^q(\mathbf{R}^d)} dt + \int_0^T \int_0^t \|\chi e^{i(s-t)P} f_n\|_{L^q(\mathbf{R}^d)} ds dt.$$

Since the support of  $u_n$  concentrates in 0, with a volume of order  $(\log \lambda_n / \lambda_n)^d$ , and  $\chi$  does not vanish in 0, the left term of this inequality is, using (14) and taking  $n$  large enough, greater than:

$$\frac{1}{C} \int_0^T \|u_n\|_{L^q(\mathbf{R}^d)} dt \geq \frac{1}{C} \left( \frac{\log \lambda_n}{\lambda_n} \right)^{d(1/q-1/q_0)} \|u_n\|_{L^{q_0}(\mathbf{R}^d)}.$$

Using the negation of (9), the right term is dominated by:

$$\|u_n\|_{L^{q_0}(\mathbf{R}^d)} + \|f_n\|_{L^{q_0}(\mathbf{R}^d)}.$$

Hence we obtain, using (8):

$$\left( \frac{\lambda_n}{\log \lambda_n} \right)^{d(1/q_0-1/q)} \|u_n\|_{L^{q_0}} = O(\|u_n\|_{L^{q_0}} + \lambda_n^{-N}),$$

which leads to the announced contradiction since  $N \geq 1$  and  $q > q_0$ , the norm of  $u_n$  in  $L^{q_0}$  being greater than 1 by (7).

By Sobolev inequalities, (9) implies (10), taking in (9)  $q_0 = 2$  and  $q > 2$  close enough to 2. So (10) holds.

Let us assume that (11) is not true. Then we get, by (15):

$$\begin{aligned} \|\chi U_n(T)\|_{L^q(\mathbf{R}^d)} &\leq C \|u_n\|_{L^{q'}(\mathbf{R}^d)} + \int_0^T \|\chi e^{-iT P} e^{is P} f_n\|_{L^1(\mathbf{R}^d)} ds \\ &\leq C \left\{ \|u_n\|_{L^{q'}(\mathbf{R}^d)} + \int_0^T \|e^{is P} f_n\|_{L^{q'}(\mathbf{R}^d)} ds \right\} \\ &\leq C \left\{ \|u_n\|_{L^{q'}(\mathbf{R}^d)} + T \|f_n\|_{L^2(\mathbf{R}^d)} \right\}. \end{aligned}$$

To obtain the second inequality we have bounded, up to a multiplicative constant, the  $L^{q'}$  norm on the support of  $f_n$  by the  $L^2$  norm, and we have used the fact that  $e^{is P}$  is an isometry on  $L^2$ . The contradiction is again a simple consequence of (14) and the fact that  $q' < q$ .

We shall now prove (12). Otherwise, we would have by (15):

$$(16) \quad \int_0^T \|U_n(t)\|_{L^q(\mathbf{R}^d)} dt \leq C \left\{ \|u_n\|_{D(P^{\sigma/2})} + \int_0^T \int_0^t \|e^{i(s-t)P} f_n\|_{D(P^{\sigma/2})} ds dt \right\}.$$

Of course,  $e^{i(s-t)P}$  maps isometrically the domain of  $P^{\sigma/2}$  onto itself. We have:

$$\|f_n\|_{D(P^{1/2})}^2 = \int |\nabla f_n(x)|^2 dx + \int (1 + V(x)) |f_n(x)|^2 dx.$$

On the support of  $f_n$ , as on that of  $u_n$ ,  $|x| \geq \frac{\log \lambda_n}{C \lambda_n}$ . Using the superior bound of  $V$  in (5), we get, on the support of  $f_n$ :

$$|V(r)| \leq C \frac{\lambda_n^2}{(\log \lambda_n)^2} (\log \log \lambda_n - \log \lambda_n)^2 \leq C \lambda_n^2.$$

It follows using the bound (8) on the semi-norms of  $f_n$  and  $N \geq 2$  that:

$$(17) \quad \|f_n\|_{D(P^{1/2})} \leq C (\|\nabla f_n\|_{L^2} + \lambda_n \|f_n\|_{L^2}) \leq C \lambda_n^{-1}.$$

Furthermore, the equation  $Pu_n - \lambda_n^2 u_n = f_n$  implies, using once again the bounds (7) and (8) on the norms of  $u_n$  and  $f_n$ :

$$\begin{aligned} \|u_n\|_{D(P^{1/2})}^2 &\leq \|f_n\|_{L^2} \|u_n\|_{L^2} + \lambda_n^2 \|u_n\|_{L^2}^2 \\ &\leq C\lambda_n^2 \|u_n\|_{L^2}^2. \end{aligned}$$

By interpolation on the norms of the left hand side, and since  $0 \leq \sigma \leq 1$ , we get:

$$(18) \quad \|u_n\|_{D(P^{\sigma/2})} \leq C\lambda_n^\sigma \|u_n\|_{L^2}.$$

According to (17), (18) and the inequality (16):

$$\|u_n\|_{L^q} \leq C\lambda_n^\sigma \|u_n\|_{L^2} + o(1).$$

Hence, with (14):

$$\left( \frac{\lambda_n}{\log \lambda_n} \right)^{d/2-d/q} \|u_n\|_{L^2} \leq C\lambda_n^\sigma \|u_n\|_{L^2} + o(1),$$

which is absurd since  $\frac{d}{2} - \frac{d}{q} > \sigma$ .

**Remark.** To get similar result on the wave equation (13), replace  $e^{-i\lambda_n^2 t} u_n$  by  $\sin(\lambda_n t) u_n$ , which is an approximate solution of (13).

### 3. PROOF OF THE THEOREM.

Denote by  $r$  the euclidian norm of  $x$  and by  $'$  the radial derivative  $\frac{d}{dr}$ . We would like to find radial functions  $V, f_n, u_n$  such that:

$$(19) \quad f_n(r) = -u_n''(r) - \frac{d-1}{r} u_n'(r) + V(r)u_n(r) - \lambda_n^2 u_n(r),$$

with  $f_n$  small and  $\lambda_n$  diverging to infinity. We shall first change functions to get rid of the first-order derivative in this equation. Set:

$$(20) \quad u_n = r^{-\frac{d-1}{2}} v_n, \quad f_n = r^{-\frac{d-1}{2}} g_n, \quad W = V + \frac{d^2 - 4d + 3}{4r^2}.$$

Thus (19) becomes:

$$(19') \quad g_n = -v_n'' + Wv_n - \lambda_n^2 v_n.$$

Let:

$$y(s) = e^{-\sqrt{s^2+1}}, \quad b(s) = -\frac{1}{(s^2+1)^{3/2}} + \frac{s^2}{s^2+1},$$

which are  $C^\infty$  solutions of the equation on  $\mathbf{R}$ :

$$(21) \quad -y''(s) + b(s)y(s) = 0$$

$$(22) \quad y(s) > 0, \forall j \in \mathbb{N}, \quad |y^{(j)}(s)| \leq C_j e^{-|s|}$$

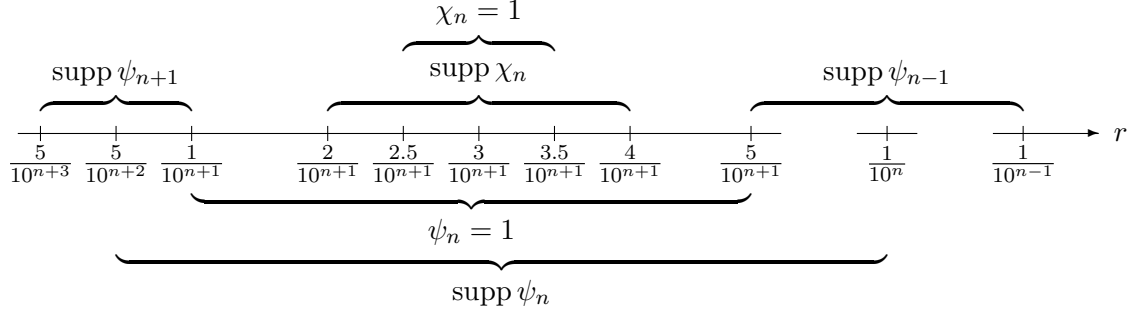
$$(23) \quad |b(s)| \leq 1.$$

We shall write:

$$y_{\omega,a}(r) = y(\omega(r-a)), \quad b_{\omega,a}(r) = \omega^2 b(\omega(r-a)),$$

where  $a$  and  $\omega$  are two real parameters. We have:

$$(21') \quad -y_{\omega,a}'' + b_{\omega,a} y_{\omega,a} = 0;$$


 FIGURE 1. Cutoff functions near  $3/10^{n+1}$ 

Let  $q(\lambda)$  be a strictly increasing positive function defined in a neighborhood of  $+\infty$  such that:

$$(24) \quad \lim_{\lambda \rightarrow +\infty} q(\lambda) = +\infty$$

$$(25) \quad \lim_{\lambda \rightarrow +\infty} \frac{q(\lambda)}{\lambda} = 0.$$

Let  $(\lambda_n)_{n \geq n_0}$  be the sequence defined by the equations:

$$10^n = q(\lambda_n),$$

so that  $\lambda_n$  diverges faster to  $+\infty$  than  $10^n$ .

Choose a cutoff function  $\chi: \chi \in C_0^\infty(]-1, 1[)$ ,  $\chi = 1$  on  $\left[-\frac{1}{2}, \frac{1}{2}\right]$ .

Set:

$$\begin{aligned} \psi_n(r) &= \chi(10^n r) - \chi(10^{n+1} r), \\ \chi_n(r) &= \chi\left(10^{n+1} \left(r - \frac{3}{10^{n+1}}\right)\right), \end{aligned}$$

The  $\psi_n$ 's form a partition of unity near 0. The support of each  $\chi_n$  is included in that of  $\psi_n$ , away from those of the  $\psi_j$ 's,  $j \neq n$  (see the figure):

$$(26) \quad \begin{aligned} \text{supp } \psi_n &\subset \left[\frac{5}{10^{n+2}}, \frac{1}{10^n}\right] \\ \frac{1}{10^{n+1}} \leq r \leq \frac{5}{10^{n+1}} &\Rightarrow \psi_n(r) = 1 \\ \sum_{n \geq n_0} \psi_n(r) &= \chi(10^{n_0} r) \end{aligned}$$

$$(27) \quad \text{supp } \chi_n \subset \left[\frac{2}{10^{n+1}}, \frac{4}{10^{n+1}}\right] \subset \{\psi_n = 1\}$$

$$(28) \quad \frac{25}{10^{n+2}} \leq r \leq \frac{35}{10^{n+2}} \Rightarrow \chi_n(r) = 1.$$

FIGURE 2. The potential  $W$ 

We shall denote by  $y_n$  and  $b_n$  the following functions:

$$(29) \quad y_n(r) = y_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = y\left(\frac{\lambda_n}{2}\left(r - \frac{3}{10^{n+1}}\right)\right)$$

$$(30) \quad b_n(r) = b_{\frac{\lambda_n}{2}, \frac{3}{10^{n+1}}}(r) = \frac{\lambda_n^2}{4} b\left(\frac{\lambda_n}{2}\left(r - \frac{3}{10^{n+1}}\right)\right).$$

Each of the function  $-b_n$  may be seen as a potential well which concentrates (by equation (21')) the energy of  $y_n$  near  $3/10^{n+1}$ . We shall construct  $W$  so that the equation  $g_n(r) = 0$  is exactly  $-v_n'' + b_n v_n = 0$  on a small interval (including the support of  $v_n$ ) around the point  $3/10^{n+1}$ . The size of this interval will be of the same order as  $10^{-n}$ , smaller than the equation parameter  $\omega = \lambda_n/2$ . We shall choose  $v_n$  as a cutoff of  $y_n$ . The exponential decay of  $y$ , combined with the difference of order between these two scales will induced small enough error terms for (7) and (8) to hold.

Set (see the figure):

$$(31) \quad W(r) = \sum_{n \geq n_0} \psi_n(r) (b_n(r) + \lambda_n^2)$$

$$(32) \quad v_n(r) = \alpha_n y_n(r) \chi_n(r),$$

where the  $\alpha_n$ 's are strictly positive constants yet to be determined. The functions  $g_n$ ,  $u_n$ ,  $f_n$  and the potential  $V$  are thus defined by (19') and (20).

**Lemma 1.** *The following inequalities hold:*

$$(33) \quad \forall j \in \mathbb{N}, \exists C > 0, \left\| \frac{d^j f_n}{dr^j} \right\|_{L^\infty(\mathbf{R}^d)} \leq C \alpha_n \lambda_n^{1+j} (q(\lambda_n))^{\frac{d+1}{2}} e^{-\frac{\lambda_n}{20q(\lambda_n)}}$$

$$(34) \quad \|u_n\|_{L^1(\mathbf{R}^d)} \geq \frac{\alpha_n}{C} \lambda_n^{-1} (q(\lambda_n))^{-\frac{d-1}{2}}.$$

*Proof.* According to (27), on the support of  $\chi_n$ ,  $W$  is equal to  $b_n + \lambda_n^2$ . Hence:

$$(35) \quad g_n(r) = -v_n''(r) + b_n(r)v_n(r).$$

Thus, using the equation on  $y_n$  given by (21'):

$$f_n(r) = r^{-\frac{d-1}{2}} g_n(r) = r^{-\frac{d-1}{2}} \alpha_n (y_n(r) \chi_n''(r) + 2y_n'(r) \chi_n'(r)).$$

The derivative of  $f_n$  of order  $j$  is then of the form:

$$(36) \quad \frac{d^j f_n}{dr^j} = \sum_{j_1+j_2+j_3=j+1} \beta_{j_1, j_2, j_3} \frac{d^{j_1}}{dr^{j_1}} y_n \frac{d^{j_2}}{dr^{j_2}} \chi_n' \frac{d^{j_3}}{dr^{j_3}} r^{-\frac{d-1}{2}}.$$

On the support of  $\chi_n'$ :

$$\left| r - \frac{3}{10^{n+1}} \right| > \frac{1}{10^{n+1}} = \frac{1}{10q(\lambda_n)}.$$

So according to the bounds (22) of  $y$  and its derivatives:

$$\begin{aligned} |y_n^{(j_1)}(r)| &\leq C \lambda_n^{j_1} e^{-\frac{\lambda_n}{2}} \left| r - \frac{3}{10^{n+1}} \right| \\ &\leq C \lambda_n^{j_1} e^{-\frac{\lambda_n}{20q(\lambda_n)}} \end{aligned}$$

Furthermore,

$$\begin{aligned} |\chi_n^{(j_2+1)}| &\leq C (q(\lambda_n))^{j_2+1} \leq C \lambda_n^{j_2} q(\lambda_n) \\ \left| \left( \frac{d}{dr} \right)^{j_3} \left( r^{-\frac{d-1}{2}} \right) \right| &\leq C (q(\lambda_n))^{\frac{d-1}{2}+j_3} \leq C \lambda_n^{j_3} (q(\lambda_n))^{\frac{d-1}{2}}, \end{aligned}$$

for on the support of  $\chi_n$ ,  $r \geq \frac{1}{10^{n+1}}$ . These three inequalities, together with (36), implies (33).

We shall now prove (34). By the definition of  $y_n$ :

$$\frac{\lambda_n}{2} \left| r - \frac{3}{10^n} \right| \leq \frac{1}{2} \Rightarrow y_n(r) \geq m = \sup_{|s| \leq \frac{1}{2}} |y(s)|.$$

Furthermore, if  $r$  is as above, and  $n$  big enough, then  $\chi_n(r) = 1$  and so:

$$u_n(r) = r^{-\frac{d-1}{2}} v_n(r) = r^{-\frac{d-1}{2}} \alpha_n y_n(r)$$

Hence:

$$\begin{aligned} \|u_n\|_{L^1} &\geq m\alpha_n \int_{\lambda_n |r - \frac{3}{10^r}| \leq 1} r^{-\frac{d-1}{2}} r^{d-1} dr \\ &\geq \frac{\alpha_n}{C} (10^{-n})^{\frac{d-1}{2}} \lambda_n^{-1}. \end{aligned}$$

□

Choose  $M > 1$  and set:

$$(37) \quad q(\lambda) = \frac{\lambda}{M \log \lambda}.$$

The positive function  $q$  is strictly increasing for big  $\lambda$ 's and satisfies (24) and (25). In addition, lemma 1 implies, bounding from above  $q(\lambda_n)$  by  $\lambda_n$ :

$$\begin{aligned} \left\| \frac{d^j f_n}{dr^j} \right\|_{L^\infty} &\leq C\alpha_n \lambda_n^{\frac{d+3}{2} + j - \frac{M}{20}} \\ \|u_n\|_{L^1} &\geq \frac{\alpha_n}{C} \lambda_n^{\frac{d+1}{2}}, \end{aligned}$$

with new constants  $C$ , which may depend on  $M$ . So the conditions (7) and (8) of the theorem are satisfied if the constants  $M$  and  $\alpha_n$  are well chosen. The support of  $u_n$  is that of  $\chi_n$ , which is of the desired form:

$$\left\{ c_1 \frac{\log \lambda_n}{\lambda_n} \leq r \leq c_2 \frac{\log \lambda_n}{\lambda_n} \right\},$$

if the support of  $\chi$  is taken to be a segment.

The assertion (5) on the potential remains to be checked. We have the following approximation of the inverse function of  $q$ :

**Lemma 2.** *Let  $q$  be defined by (37). Then:*

$$q(\lambda) \log(q(\lambda)) \sim \frac{\lambda}{M}, \quad \lambda \rightarrow +\infty.$$

*Proof.*

$$\begin{aligned} q(\lambda) \log q(\lambda) &= q(\lambda) \log \left( \frac{\lambda}{M \log \lambda} \right) \\ &= q(\lambda) (\log \lambda - \log \log \lambda - \log M) \\ &\sim q(\lambda) \log \lambda, \end{aligned}$$

when  $\lambda$  goes to infinity. □

On the support of  $\psi_n$ :

$$\begin{aligned} \frac{1}{20r} &\leq q(\lambda_n) \leq \frac{1}{r} \\ \frac{1}{20r} |\log r + \log 20| &\leq q(\lambda_n) \log(q(\lambda_n)) \leq \frac{1}{r} |\log r|. \end{aligned}$$

(We used that  $s \mapsto s \log s$  is an increasing function for  $s > e^{-1}$ ). Using lemma 2 and taking  $r$  close enough to 0 we get:

$$C^{-1} \frac{1}{r} |\log r| \leq \lambda_n \leq C \frac{1}{r} |\log r|.$$

Thus, by the definition (31) of  $W$ ,  $|b_n|$  being bounded from above by  $\lambda_n^2/4$ :

$$C^{-1} \frac{|\log r|^2}{r^2} \sum_{n \geq n_0} \psi_n(r) \leq W(r) \leq C \frac{|\log r|^2}{r^2} \sum_{n \geq n_0} \psi_n(r),$$

which implies, with (26), the inequality (5) on the potential  $W$ . Taking a greater  $C$  if necessary, we get that  $V$  satisfies the same inequality.

- Remarks.**
- To get quasi-modes of infinite order, we would have taken  $q(\lambda) = \frac{\lambda}{(\log \lambda)^{1+\varepsilon}}$  and suitably modified lemma 2.
  - In the preceding construction, we could have replaced  $y$  and  $b$  by any non-trivial solutions of the equation (21) satisfying the bounds (22) and (23).

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