Hertz potentials, peeling, and the Cauchy problem

Jérémie Joudioux
Max-Planck-Institut für Gravitationsphysik
Albert-Einstein-Institut
Golm

Joint work with Lars Andersson and Thomas Bäckdahl.

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Introduction

- Context: Study of the nonlinear stability of Kerr black holes.
- Previous result: Minkowski nonlinear stability ('91), relies heavily on decay result for higher spin fields ('89) on flat background.
- Problem: develop new methods to understand the the decay properties of fields.
- Question: develop alternate methods to study the asymptotics of higher spin fields
- Exploit symmetries of the spacetime/higher spin fields (Maxwell fields and linearized gravity).
- the structure of potentials (Hertz, Debye, etc) strongly tight to the structure of the space-time.
Introduction : Hertz potentials

- Penrose (63’) : representations of massless spin-s fields on flat space-time : local representation by a potential of order $2s$, satisfying a wave equation.
- Penrose proved peeling from a decay assumption on $\chi$.
- (Cohen-Kegeles 76) : on Kerr black holes :
  \[ F = \overline{d\delta}G, \ G \text{ solution of a wave equation} \]

is an uncharged solution of the Maxwell equations.
- Conjecture : Any Maxwell fields can be written as :
  \[ F = F_{\text{Coulomb}} + \overline{d\delta}G \]
- Conjecture : in this situation, $\overline{d\delta}G$ radiates/decays, under suitable assumptions.
Today’s talk’s framework and purpose

- Background: flat space-time.
- Cauchy problem for massless spin-s fields of arbitrary spin but especially Maxwell (spin 1) and linearized gravity (spin 2).
- Construct a potential satisfying a wave equation, whose initial data lie in a Sobolev space insuring good decay properties.
- Deduce decay/peeling properties.
- Important result: Christodoulou-Klainerman ’89 on linear fields.
Table of contents

1 Standard decay results for the scalar wave equation

2 Construction of potentials

3 Decay of linear fields using potentials
Scalar fields on flat background

• Restrict our attention to fields on Minkowski space:

\((\mathbb{R}^4, dt^2 - dx^2 - dy^2 - dz^2)\)

• Consider the Cauchy problem for the wave equation:

\[
\begin{aligned}
    \Box \phi &= 0 \\
    \phi|_{t=0} &= f \in H^k_\sigma(\mathbb{R}^3) \\
    \partial_t \phi|_{t=0} &= g \in H^{k-\frac{1}{2}}_\sigma(\mathbb{R}^3)
\end{aligned}
\]

• Purpose: How does this field decay?
Background: flat space-time.

Obtained by energy estimates (Klainerman 83-87, $\sigma = -\frac{3}{2}$) or by conformal compactification (Penrose 65, stronger assumptions on the initial data, $\sigma = -3$).

There exist more general works, for arbitrary weights (Asakura '86, d’Ancona-Georgiev-Kubo '01, Szpak '08).

Obtain decay estimates in two directions:
- Interior decay: along time directions ($t > 3r$);
- Exterior decay: along null directions ($\frac{t}{3} < r < 3t$).
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Standard decay results for the scalar wave equation

Decay of solutions of the linear wave equation

**Theorem (Klainerman)**

Let $s_0 \geq 2$. Let $u$ be a solution of the wave equation with initial date in $H^{s_0}_{\frac{3}{2}}(\mathbb{R}^3) \times H^{s_0 - \frac{1}{2}}_{\frac{5}{2}}(\mathbb{R}^3)$. Then

1. for $t > 3r$
   
   $$|\phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0}}{< t >^{\frac{3}{2}}},$$

2. for $\frac{r}{3} < t < 3r$
   
   $$|\phi(t, x)| \leq C \frac{||\phi(0)||_{-\frac{3}{2}, s_0}}{< u >^{\frac{1}{2}} < v >^{1}}.$$
Theorem (Klainerman)

Let $s_0 \geq 2$, $(j, k, l) \in \mathbb{N}^3$. Let $u$ be a solution of the wave equation with data in $H_{-3/2}^{s_0+j+k+l}$. Then

1. For $t > 3r$

$$|\nabla^j \phi(t, x)| \leq C \frac{||\phi(0)||_{-3/2, s_0+j}}{< t >^{3/2+j}},$$

2. For $\frac{r}{3} < t < 3r$:

$$|\partial^j_u \partial^k_v \nabla^l_{S_2^r} \phi(t, x)| \leq C \frac{||\phi(0)||_{-3/2, s_0+j+k+l}}{< u >^{1/2+j} < v >^{1+k+l}},$$

$u = t - r$ and $v = t + r$. 
Decay result for arbitrary weights

- For arbitrary $\sigma \notin \mathbb{Z}$?
- $\phi|_{t=0} = f \in H^k_{\sigma}$ and $\partial_t \phi|_{t=0} = g \in H^{k-1}_{\sigma-1}$, $k \geq 3$:
  \[ f(x) \leq <r>^\sigma \|f\|_{2,\sigma} \quad \text{and} \quad g(x) \leq <r>^{\sigma-1} \|g\|_{2,\sigma-1} \]
- Integral representation:
  \[
  \phi(t, x) = \frac{1}{4\pi} \left( \int_{S^2} t (g(x + t\omega) + \partial_\omega f(x + t\omega)) + f(x + t\omega)d\mu_{S^2} \right)
  \]
- Asymptotic behavior is given by:
  \[
  J_\sigma = \int_{S^2} <|x + t\omega|^\sigma > d\mu_{S^2}.
  \]
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Standard decay results for the scalar wave equation

Decay result for arbitrary weights

\[ J_{\sigma} = \begin{cases} 
8\pi \left( \frac{<u>^{2+\sigma} - <v>^{2+\sigma}}{(2 + \sigma)(<u>^2 - <v>^2)} \log \left( \frac{<u>}{<v>} \right) \right) & \text{if } \sigma \neq -2 \text{ and } <u> \neq <v>, \\
8\pi \left( \frac{<u>^2 - <v>^2}{<u>^2 - <v>^2} \right) & \text{if } \sigma = -2 \text{ and } <u> \neq <v>, \\
4\pi <v>^{\sigma} & \text{if } <u> = <v>. 
\end{cases} \]

For the full solution, combine \( J_{\sigma} \) and \( J_{\sigma-1} \), hence the discussion arises on \( \sigma = -1 \).

For higher order derivatives, one use commutations with \( \square \).
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Standard decay results for the scalar wave equation

Decay for arbitrary weights

**Proposition**

If \((f, g)\) in \(H^m_\sigma \times H^{m-1}_{\sigma-1}\), \(m \geq j + k + l + 3\), one denotes:

\[
l_\sigma = \left\| (f, g) \right\|_{H^j_\sigma + k + l + 3 \times H^{j+k+l+2}_{\sigma-1}}
\]

then:

\[
\left\| \partial_u^k \partial_v^l \nabla^m S_2 \phi \right\| \leq C_{l_\sigma} \begin{cases} 
\langle u \rangle^{1+\sigma-k} \langle v \rangle^{-1-l-m} & \text{if } \sigma < k-1 \\
\log \langle v \rangle - \log \langle u \rangle & \text{if } \sigma = k-1 \\
\langle v \rangle^{l+m} (\langle v \rangle - \langle u \rangle) & \text{if } \sigma > k-1,
\end{cases}
\]

Rem : The estimates are sharp.
Problem

- Give a proper analytic framework to Penrose’s representation of massless fields of spin $2s$:
  \[
  \phi_{A...F} = \nabla_{AA'} \cdots \nabla_{FF'} \xi^{A'...F'}, \text{where } \square \xi = 0.
  \]

- Proper analytic representation: Cauchy problem for the field to a Cauchy problem for the potential + control of the norm of the initial data of the potential.

- Cases of interest: Maxwell and linearized gravity on flat background.

- Methods: elementary elliptic theory.
Hertz potentials, peeling, and the Cauchy problem

Construction of potentials

Maxwell equations

- Geometric background: Minkowski background \((\mathbb{R}^4, \eta)\).
- Consider the Faraday (skew-symmetric) tensor (2-form) : \(F\).
- Link with electric and magnetic fields (1-forms on \(\mathbb{R}^3\) ) :
  \[ T^a = (1, 0, 0, 0) : \]
  \[ E = F(T, \bullet) \text{ and } B = (\star F)(T, \bullet) \]
  where \(\star F\) is the Hodge dual.
- Maxwell equations :
  \[ dF = 0 \text{ and } \delta F = 0 \]
  \[ \nabla_{[a}F_{bc]} = 0 \text{ and } \nabla^a F_{ab} = 0 \]
Cauchy problem for the Maxwell equations

- Hyperbolic system of order 1 with 6 real unknowns with geometric constraints on the initial data.

\[
\begin{align*}
(\bar{d} + \bar{\delta})F &= 0 \\
F|_{t=0} &\in H^{k}_o(\mathbb{R}^3, \Lambda^2)
\end{align*}
\]

- Geometric constraints on the initial data:

\[
D^a E_a = D^a B_a = 0
\]

- Purpose: construct initial data for \( G \) such that \( F \) of the form:

\[
F = \bar{d}\delta G
\]
Construction of a potential for the Maxwell field

- Assume $F = \overline{d\delta} G$.
- Restrict to $t = 0$:
  \[
  E = -\delta dH - \delta \star \partial_t K \\
  B = -\delta dK + \delta \star \partial_t H
  \]
- Solve for a given set of initial data with:
  \[
  H = K = 0.
  \]
- Take $E, B \in H^k_\sigma$ in the image of $\Delta$:
  \[
  E = (d\delta + \delta d) \tilde{H} \\
  B = (d\delta + \delta d) \tilde{K}
  \]
  with $\tilde{H}, \tilde{K} \in H^{k+2}_{\sigma+2}$. 
Construction of a potential

• Use the geometric constraints:

\[ \delta E = 0 \quad \Rightarrow \quad E = \delta \left( d\tilde{H}(\text{sth}) \right) \]
\[ \delta B = 0 \quad \Rightarrow \quad B = \delta \left( d\tilde{K}(\text{sth}) \right) \]

• Take as initial data for the potential: \((0, -\star d\tilde{H})\) and \((0, \star d\tilde{K})\).

• Conditions to admit a potential:

\[ E, B \in H^k_\sigma(\Lambda^1) \perp \text{Ker}(\Delta) \cap L^2_{-3-\sigma}(\Lambda^1) \]

• Claim: \(E, B\) satisfying the constraints are automatically orthogonal to the elements of \(L^2_{-3-\sigma}(\Lambda^1)\) satisfying:

\[ d\omega = 0. \]
Construction of a potential

- for $\delta = -\frac{7}{2}$, the potential has data lying in $H_{-\frac{3}{2}}$ and
  \[ \text{Ker}(\Delta) \cap L^2_{\frac{1}{2}}(\Lambda^2) = \{ \alpha_i dx^i | \alpha_i \in \mathbb{R} \}; \]

  \[ \text{CK} \quad \text{for} \ \delta = -\frac{5}{2}, \text{the potential has data lying in} \ H_{-\frac{1}{2}} \text{ and} \]
  \[ \text{Ker}(\Delta) \cap L^2_{\frac{1}{2}}(\Lambda^2) = \{ 0 \}; \]

- In both cases, the orthogonality condition is satisfied.
- Constraints + data in $H^k_{\sigma}$ ($\sigma = -\frac{5}{2}, -\frac{7}{2}$) $\Rightarrow$ automatically orthogonal to the cokernel.
Existence of a potential for Maxwell fields

**Proposition**

Let $\sigma$ in $\{-\frac{5}{2}, -\frac{7}{2}\}$ and $s_0 \geq 2$. Let $E_0, B_0$ be two solutions of the constraints in $H^s_{\sigma}$. Then there exist two 2-forms $(G_0, G_1)$ in $H^{s_0+2}_{\sigma+2} \times H^{s_0+1}_{\sigma+1}$ such that:

$$\|G_0\|_{s_0+2,\sigma+2}^2 + \|G_1\|_{s_0+1,\sigma+1}^2 \leq C\|F_0\|_{s_0,\sigma}^2$$

and:

$$F = \overline{d\delta} G$$

where $G$ is the solution of the wave equation $(\overline{d\delta} + \delta d)G = 0$ with initial data $(G_0, G_1)$. 
Linearized gravity – Spinor version

- \( W_{abcd} \), a 4-tensor satisfying the symmetries of the Weyl spinor.
- Consider the Cauchy problem:
  \[
  \begin{cases}
    \nabla^a W_{abcd} = 0 \\
    W_{abcd} = \psi_{abcd} \in H^k + \text{constraints}
  \end{cases}
  \]
- Introduce \( E, B \):
  \[
  \begin{aligned}
    E_{cd} &= T^a T^b W_{abcd} \\
    D^a E_{ab} &= 0 \\
    B_{cd} &= T^a T^b (\star W_{abcd}) \\
    D^a B_{ab} &= 0
  \end{aligned}
  \]
- Hyperbolic system of order 1 of 10 unknowns.
- Unfortunately, no simple tensor notations:
  \[
  W = \nabla \nabla \underbrace{\nabla \nabla \xi}_{\text{Lanczos potential}}.
  \]
Linearized gravity – Spinor version

- $\phi_{ABCD}$, a totally symmetric spinor.
- Consider the Cauchy problem:

$$\begin{cases}
\nabla^{AA'} \phi_{ABCD} = 0 \\
\phi_{ABCD} = \psi_{ABCD} \in H^k + \text{constraints } D^{AB} \psi_{ABCD} = 0
\end{cases}$$

- Hyperbolic system of order 1 of 5 complex unknowns.
- In spinors, the potential writes:

$$\phi_{ABCD} = \nabla_{AA'} \nabla_{BB'} \nabla_{CC'} \nabla_{DD'} \xi^{A'B'C'D'}.$$  

Bergman potential

Lanczos potential

- Unfortunately, no simple tensor notations: $W = F_4(\xi)$
Sketch of the proof

- Principle of the proof is the same except that one has to work with $\Delta^2$.
- Only important change: orthogonality condition on $E_{ab}$ and $B_{ab}$:

**Proposition**

$E_{ab}$, satisfying the constraints, is automatically orthogonal to the elements $\omega_{ab}$ of Ker($\Delta^2$) $\cap$ $L^2_{-3-\delta}(S^2)$, which satisfy:

$$\mathcal{R}(\omega_{ab}) = 0$$

where $\mathcal{R}$ is the linearized Cotton-York tensor (3rd order differential operator).
Conformal rigidity: the deformation of a metric \( g_0, \{ g_t \}_t \), is conformally rigid iff there exist a family of diffeomorphisms \( \phi_t \) and functions \( u_t \) such that:

\[
\phi_t^* g_0 = e^{u_t} g_t
\]

with \( \phi_0 = \text{Id} \) and \( u_0 = 0 \).

The conformal Killing equation:

\[
L_X g_0 - \frac{1}{3} \text{Tr}(L_X g_0) g_0 = h \quad \text{or} \quad 2D_{(AB}X_{CD)} = h_{ABCD}
\]

can only be integrated provided that:

\[
0 = \epsilon_{abcd} \mathcal{R}(h)_{dc} = 2D_{[a}\sigma_{b]}c \quad \text{where}
\sigma_{ab} = D_{(a}D^c h_{b)c} - \frac{1}{2} \Delta h_{ab} - \frac{1}{4} g_{ab} D^c D^d h_{cd}.
\]

\( \mathcal{R} \) is the linearized Cotton-York tensor.
De Rham and Gasqui-Goldschmidt's complexes

- Previous works: Gasqui-Golschmidt '84; Beig '97
- Solving

\[ df = \star \omega \left( \omega \in \Lambda^1 \right) \text{ or } L_X g_0 - \frac{1}{3} \text{Tr}(L_X g_0)g_0 = h \]

requires that the lhs satisfy constraints.

- Constraints are solved by the differential complexes:

\[
\begin{align*}
C^\infty(M, \mathbb{R}) & \xrightarrow{d} \Lambda^1 \xrightarrow{\star d} \Lambda^1 \xrightarrow{\delta} C^\infty(M, \mathbb{R}) \\
\Lambda^1(M) & \xrightarrow{L} S^2_0(M, g) \xrightarrow{\mathcal{R}} S^2_0(M, g) \xrightarrow{\delta^2} \Lambda^1(M)
\end{align*}
\]

L : conformal Killing operator, \( \delta_2 \) : divergence on 2 tensors.
Proposition

Let $\sigma$ in $\{-\frac{11}{2}, -\frac{9}{2}, -\frac{7}{2}\}$ and $s_0 \geq 2$.
Let $\psi_{ABCD}$ be a solution to the constraints in $H_{s_0}^{\sigma}$.
Then there exists $(\xi_0, \xi_1)$ in $H_{\sigma+4}^{s_0+4} \times H_{\sigma+3}^{s_0+3}$ such that :

$$\|\xi_0\|_{s_0+2,\sigma+4}^2 + \|\xi_1\|_{s_0+1,\sigma+3}^2 \leq C\|\psi\|_{s_0,\sigma}^2$$

and

$$\phi_{ABCD} = \nabla AA' \nabla BB' \nabla CC' \nabla DD' \xi^{A'B'C'D'}$$

where $\xi$ is the solution of the wave equation $\Box \xi = 0$ with initial data $(\xi_0, \xi_1)$. 
Purpose: Study the asymptotic behavior of spin $s$ fields satisfying the Dirac equation on flat background by methods which could be extended to Kerr background.


Here: derive the same kind of decay result using representation of fields using potentials by reducing the tensor equation to a scalar wave equation.

Methods introduced by Penrose in ’61/’65 (spin lowering and spin raising process).
### Peeling for the Maxwell equations

<table>
<thead>
<tr>
<th>Weight – ID</th>
<th>$-\frac{7}{2}$</th>
<th>$-\frac{5}{2}$ (ABJ)</th>
<th>$-\frac{5}{2}$ (CK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight – ID potential</td>
<td>$-\frac{3}{2}$</td>
<td>$-\frac{1}{2}$ (ABJ)</td>
<td>X</td>
</tr>
</tbody>
</table>

#### Interior decay $t > 3r$

<table>
<thead>
<tr>
<th></th>
<th>$t^{-\frac{7}{2}}$</th>
<th>$t^{-\frac{5}{2}}$</th>
</tr>
</thead>
</table>

#### Exterior decay $\frac{t}{3} < r < 3t$

<table>
<thead>
<tr>
<th></th>
<th>$u^{-\frac{5}{2}} v^{-1}$</th>
<th>$u^{-\frac{3}{2}} v^{-1}$</th>
<th>$u^{-\frac{3}{2}} v^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(\partial_v, e_1), \phi_2$</td>
<td>$u^{-\frac{1}{2}} v^{-2}$</td>
<td>$u^{-\frac{1}{2}} v^{-2}$</td>
<td>$u^{-\frac{1}{2}} v^{-2}$</td>
</tr>
<tr>
<td>$F(\partial_v, \partial_u), F(e_1, e_2), \phi_1$</td>
<td>$u^{-\frac{1}{2}} v^{-3}$</td>
<td>$r^{-\frac{5}{2}}$</td>
<td>$r^{-\frac{5}{2}}$</td>
</tr>
<tr>
<td>$F(\partial_u, e_1), \phi_0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One can also derive these components to complete the peeling result. The result is still the same as the one as CK.
In the interior region, for all weight $\sigma$:

$$|\phi_{ABCD}| \lesssim \frac{1}{<t>^\sigma}.$$ 

For the weight $\sigma = -\frac{11}{2}$, the exterior decay result is:

$$|\phi_i| \lesssim \frac{1}{v^{1+4-i}u^{\frac{1}{2}+i}}.$$ 

For the weight $\sigma = -\frac{7}{2}$:

- for $i = 2, 3, 4$,

  $$|\phi_i| \lesssim \frac{1}{<v>^{1+4-i}<u>^{\frac{5}{2}+i}}.$$ 

- for $i = 0, 1$,

  $$|\phi_i| \lesssim <r>^{-\frac{7}{2}}.$$ 

Exactly the same result as CK.
Conclusion

- We recover fully Christodoulou-Klainerman results; but the mechanism is simpler and works for arbitrary spin.
- For half integer spin: should work similarly, with the extra remark that the curl is elliptic in this situation.
- Purpose: extend this to Kerr space time.
- In this context, Maxwell fields of the form $d\delta G$ are not charged.
- Hard: Need a proper elliptic theory, results on the wave equation are partially complete.
- Nonetheless: there exists another process: spin lowering, which can generate both symmetries amongst solutions and potentials, using the existence of the Killing spinor.