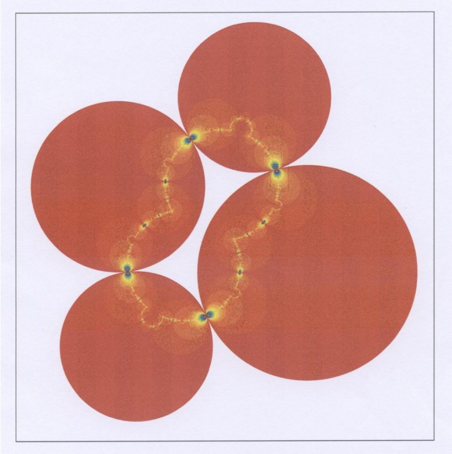

Highly complex: Möbius transformations, hyperbolic tessellations and pearl fractals

A Möbius[1] transformation[2] $f(z)=(az+b)/(cz+d)$ – with complex coefficients a,b,c,d – is an invertible conformal map from the extended complex plane into itself. Among them, we find translations, dilations, rotations as well as inversions (with respect to circles); moreover, every Möbius transformation can be composed from a number of such maps. Using the stereographic projection[3] from the Riemann sphere[4] to the extended complex plane, one may think of Möbius transformations as conformal (or biholomorphic) maps from the sphere into itself. It is worthwhile to investigate their geometric, algebraic and analytic properties; in our days, many of them can be visualized by various applets on the internet.

1À http://www-history.mcs.st-andrews.ac.uk/Biographies/Mobius.html

2À http://en.wikipedia.org/wiki/M%C3%B6bius_transformation

3À http://en.wikipedia.org/wiki/Stereographic_projection

4À http://en.wikipedia.org/wiki/Riemann_sphere

In the late 19th century, Möbius transformations played an important role in the work of Felix Klein[5] and Henri Poincaré[6] in making the hyperbolic plane and hyperbolic space – until then quite abstract inventions – more easily understandable through concrete models. Certain subgroups of the group of Möbius transformations can be interpreted as isometry groups for these models. Furthermore, discrete subgroups of these isometry groups can be used to tessellate[7] the entire hyperbolic plane by hyperbolic polygons. Several of M.C. Escher[8]'s most fascinating pictures make use of this technique.

Imagine a bunch of circles that are paired by certain Möbius transformations: inside out, outside in! Already back in the late 19th century, Klein, Robert Fricke[9] and Friedrich Schottky[10] tried to find out what happens when you iterate these maps. It is amazing how close their "fractal" hand drawings come to our days computer generated pictures that you can draw on your screen using programs on the internet. With a little experience, you can even generate very attractive pearl (!) necklaces yourself.

This talk is inspired by the beautiful book *Indra's Pearls*[11] by Mumford, Series and Wright.

5À http://www.gap-system.org/~history/Biographies/Klein.html

6À http://www-history.mcs.st-andrews.ac.uk/~history/Biographies/Poincare.html

7À http://en.wikipedia.org/wiki/Tessellation

8À http://en.wikipedia.org/wiki/M._C._Escher

9À http://en.wikipedia.org/wiki/Robert_Fricke

10À http://en.wikipedia.org/wiki/Friedrich_Schottky

11À http://www.cambridge.org/uk/catalogue/catalogue.asp?isbn=0521352533