Problem set 3: Generating Functions and the Configuration Model

Ex.1. Obtain the analytical expression of the generating function of the Poisson distribution.

Ex.2. Prove the following expression:
\[
\sum_{x} x^n (\frac{\lambda^x}{x!}) = e^{\lambda (1-e^{-\lambda})}
\]

Ex.3. Using the excess degree distribution, prove that for the configuration model, the clustering coefficient may be written as:
\[
C = \frac{1}{N} \frac{\langle k^2 \rangle - \langle k \rangle^2}{\langle k \rangle^3}
\]
where N is the size of the network and \(\langle k \rangle\), \(\langle k^2 \rangle\) are the first and second moments of the degree distribution \(p_k\) respectively.

Though this expression is formally the same that one obtains for the uniform random graph, does it lead to the same consequences concerning the dependence of C on the size of the system? Why?

Ex.4. Obtain the expression of the generating function, \(g_1(z)\) for the excess degree distribution \(q_k\) of the Poisson distribution \(p_k\).

Ex.5. Using the “power” property of generating functions prove that the generation function
\[
g^{(2)}(z) = \sum_{k=0}^{\infty} p_k^{(2)} z^k
\]
of the probability, \(p_k^{(2)}\), that a randomly chosen vertex has exactly k second neighbours, is:
\[
g^{(2)}(z) = g_0(g_1(z))
\]

Ex.6. Consider the following distribution function: \(p_0, p_1, p_2, p_3 >0\) and \(p_k = 0\ \forall \ k>3\).

a) Obtain the generating functions of \(g_0(z)\) and \(g_1(z)\) of \(p_k\) and of the excess degree distribution \(q_k\), respectively.

b) Show that the condition for a giant component to exist is: \(p_3 > p_1/3\).

c) Obtain the size of the giant component \(S\) in terms of the distribution \(p_k\).