UNIVERSITÉ de Cergy-Pontoise

Master TPA-M2

Complex Systems I

Problem set 4. Modelling real networks: Small World and Exponential Random Graphs

Ex.1. Consider a 1-d network of N sites with periodic boundary conditions. Each site has r neighbours on each side.

a) Derive the expression for the clustering coefficient.
b) Derive the expression of the average path length. Which its large N limit expression? Could this result have been estimated using a rough approximation?

Ex. 2. The small world network, Newman's approach.

A small world network may be obtained starting from the ordered lattice of the previous exercise, where we call c=2r the degree of the initial lattice, by adding s_i shortcuts to the vertex i, so that its degree is $k_i=c+ s_i$ with probability p.

For each edge of the initial lattice a shortcut is added with probability p.

Show that in the regime of low p, the degree distribution of this network is:

$$P(k) = \begin{cases} \exp\big(-p(k-c)\big)\dfrac{(cp)^{k-c}}{(k-c)!} & k \geq c \\ 0 \; otherwise \end{cases}$$

Ex.3. The scaling function obtained in mean field approach by Newman et al. (PRL **84,** 3201 (2000)) for the small world network is: $\frac{lc}{N} = f(Ncp)$, where l is the average path length, p is the probability of adding a shortcut, c is the degree of the initial 1d lattice (see previous exercise), and :

$$f(x) = \frac{2}{\sqrt{x^2+4x}} \tanh^{-1}\sqrt{\frac{x}{x+4}}$$

Show that, in the corresponding regime that you will define, it is a small world network.

NB: recall the identity: $\tanh^{-1} u = \frac{1}{2}\ln\frac{1+u}{1-u}$

Ex.4. Let us consider an undirected exponential random graph, where one specifies the degree k_i, i=1, N of each vertex (the configuration model).

a) Show that the probability p_{lm} of connecting vertices l and m is: $p_{lm} = \frac{1}{1+\exp-(\beta_l+\beta_m)}$ where the β_l with l=1, N are external control parameters.
b) Show that for the sparse graph it becomes: $p_{lm} \approx \frac{<k_l><k_m>}{2<m>}$ with <m> the average number of edges of the network.

Ex.5. Consider an undirected exponential random graph model in which the Hamiltonian takes the form: $H = \sum_{i<j}\gamma_{ij}A_{ij}$, where the γ_{ij} are external parameters.

a) Derive the expression for the free energy.
b) Show that the probability of an edge between vertices i and j is : $p_{ij} = \frac{1}{(e^{-\gamma_{ij}}+1)}$