Problem set 4. Modelling real networks: 
Small World and Exponential Random Graphs

Ex.1. Consider a 1-d network of N sites with periodic boundary conditions. Each site has r neighbours on each side.

a) Derive the expression for the clustering coefficient.
b) Derive the expression of the average path length. Which its large N limit expression?
   Could this result have been estimated using a rough approximation?

Ex.2. The small world network, Newman’s approach.

A small world network may be obtained starting from the ordered lattice of the previous exercise, where we call c=2r the degree of the initial lattice, by adding s_i shortcuts to the vertex i, so that its degree is k_i=c+s_i with probability p.

For each edge of the initial lattice a shortcut is added with probability p.

Show that in the regime of low p, the degree distribution of this network is:

\[ f(x) = \frac{2}{\sqrt{x^2 + 4x}} \tanh^{-1} \left( \frac{x}{\sqrt{x + 4}} \right) \]

Ex.3. The scaling function obtained in mean field approach by Newman et al. (PRL 84, 3201 (2000)) for the small world network is:

\[ f(l) = 2 \sqrt{l^2 + 4l} \]

Show that, in the corresponding regime that you will define, it is a small world network.

NB: recall the identity: \( \tanh^{-1} u = \frac{1}{2} \ln \frac{1+u}{1-u} \)

Ex.4. Let us consider an undirected exponential random graph, where one specifies the degree k_i, i=1, N of each vertex (the configuration model).

a) Show that the probability p_{lm} of connecting vertices l and m is:

\[ p_{lm} = \frac{1}{1 + \exp(-\beta_i \beta_m)} \]

b) Show that for the sparse graph it becomes:

\[ p_{lm} \approx \frac{<k_i><k_m>}{2\langle m \rangle} \]

with \( \langle m \rangle \) the average number of edges of the network.

Ex.5. Consider an undirected exponential random graph model in which the Hamiltonian takes the form:

\[ H = \sum_{i<j} \gamma_{ij} A_{ij} \]

where the \( \gamma_i \) are external parameters.

a) Derive the expression for the free energy.
b) Show that the probability of an edge between vertices i and j is:

\[ p_{ij} = \frac{1}{(e^{-\gamma_{ij+1}})} \]