Solution to a conjecture on the classical entropy of quantum states

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Cergy-Pontoise June 25, 2013

\(^1\)Joint work with Elliott Lieb
To quantify the inherent uncertainty of quantum states, Wehrl ('79) suggested a definition of their classical entropy based on the coherent state transform.

He conjectured that this classical entropy is minimized by states that also minimize the Heisenberg uncertainty inequality, i.e., Gaussian coherent states.

Lieb ('78) proved this conjecture and conjectured that the same holds when Euclidean Glauber coherent states are replaced by SU(2) Bloch coherent states.

This generalized Wehrl conjecture has been open for almost 35 years. I will present a short proof which is joint work with Elliott Lieb.
Outline of Talk

1. Coherent states and quantization
2. States of minimal classical entropy
3. Quantization of the sphere
4. Generalization to Quantum Channels
5. Using bosonic 2nd quantization
6. Formulation in terms of majorization
7. Inductive argument
8. Using duality
9. A recursive inequality
10. The classical limit (if time permits)
• **Classical phase space**: $\mathcal{M} = \mathbb{R}^{2n}$ position and momentum $(q, p)$.

• **Quantum description**: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$.

• **Quantization**: Function $A$ on $\mathcal{M}$ to operator $\text{Op}(A)$ on $\mathcal{H}$.

• **Pure states**\(^2\): Described by normalized $\psi \in L^2(\mathbb{R}^n)$ gives “distribution” on phase space $\Phi_\psi$ such that

\[
\langle \psi, \text{Op}(A)\psi \rangle = (2\pi)^{-n} \int \int \Phi_\psi(q,p)A(q,p)dqdp
\]

• **Weyl quantization** leads to $\Phi_\psi(q,p)$ Wigner distribution, which is **not necessarily positive**.

• Better to use **Wick** or **coherent state** quantization

\(^2\)The state is really represented by the 1-dim projection $|\psi\rangle\langle\psi|$. More general non-pure states represented by density matrices (operators): $0 \leq \rho$, $\text{Tr} \rho = 1$. 

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Coherent state quantization

**Coherent states**, i.e., states of minimal Heisenberg uncertainty

\[ f_{q,p}(x) = \pi^{-n/4} \exp\left(-\frac{(x - q)^2}{2} + ipx\right) \in L^2(\mathbb{R}^n) \]

satisfy \((x + \nabla) f_{q,p} = (q + ip) f_{q,p}\).

They define quantization map

\[ Op(A) = (2\pi)^{-n} \iint A(q,p) |f_{q,p}\rangle \langle f_{q,p}| dq dp. \]

leads to lower or covariant symbol or Husimi \(Q\)-function

\[ \Phi_\psi(q,p) = |\langle f_{q,p}|\psi\rangle|^2. \]

Then \(0 \leq \Phi_\psi(q,p) \leq 1\) and \((2\pi)^{-n} \iint \Phi_\psi(q,p) dq dp = 1\).

**Wehrl classical entropy:**

\[ S_{cl}^{cl}(\psi) = (2\pi)^{-n} \iint -\Phi_\psi(q,p) \log \Phi_\psi(q,p) \, dq dp. \]
Theorem (Lieb ’78, Conjectured by Wehrl)

States of minimal entropy are states of minimal Heisenberg uncertainty, i.e., for all $\psi$ and all $q, p$

$$S^\text{cl}(\psi) \geq S^\text{cl}(f_{q,p}).$$

Proof based on sharp Young and Hausdorff-Young inequalities\(^3\). Carlen ’91 proved “uniqueness” of minimizers.

In fact, $-t \log(t)$ may be replaced by any concave function:

Theorem (Lieb-Solovej ’12)

For all continuous concave $f : [0, 1] \rightarrow \mathbb{R}, f(0) = 0$

$$\int \int f(\Phi_{\psi}(q', p')) dq' dp' \geq \int \int f(\Phi_{f_{q,p}}(q', p')) dq' dp'$$

\(^3\)Note that both Y and HY inequalities are optimized by Gaussians
Quantization of the sphere $S^2$

**Hilbert space:** $\mathcal{H}_J = \mathbb{C}^{2J+1}$, $J = 0, 1/2, 1, 3/2, \ldots$

**Irreducible representation** of $SU(2)$ on $\mathcal{H}_J$: Lie-algebra is represented as spin operators $\mathbf{S} = (S_x, S_y, S_z)$.

**Quantization of classical phase space** $S^2$: For each $\omega \in S^2$ let the **highest weight vector** $|\omega\rangle \in \mathcal{H}_J$ be given by $\omega \cdot \mathbf{S} |\omega\rangle = J |\omega\rangle$.

**Quantization map:**

$$Op(A) = \frac{2J + 1}{4\pi} \int_{S^2} A(\omega) |\omega\rangle \langle \omega| d\omega, \quad \Phi^\infty_v(\omega) = |\langle \omega|v\rangle|^2, \ v \in \mathcal{H}_J$$

**Theorem (Lieb-Solovej ’12, Conjectured by Lieb ’78 for entropy)**

For all $J$, all concave $f : [0, 1] \rightarrow \mathbb{R}$, all $v \in \mathcal{H}_J$, and all $\omega \in S^2$

$$\int_{S^2} f (\Phi^\infty_v(\omega')) d\omega' \geq \int_{S^2} f (\Phi^\infty_\omega(\omega')) d\omega'$$

Was known for $J \leq 3/2$ and $f(t) = -t \log t$ (Schupp ’99, Scutaru ’02).
Generalization to Quantum channels

$\Phi^\infty$ is a map from a quantum state to a classical prob. distribution. We generalize to completely positive trace preserving maps, i.e., quantum channels $\Phi^k$ from operators on $\mathcal{H}_J$ to operators on $\mathcal{H}_K$, $K = J + k/2$. Use representation $\mathcal{H}_J = \bigotimes_{\text{sym}}^{2J} \mathbb{C}^2$. Define

$$\Phi^k(\rho) = \frac{2J+1}{2K+1} P_{\text{sym}}(\rho \otimes I_{\bigotimes k \mathbb{C}^2}) P_{\text{sym}}$$

Let $\Phi^k(v) = \Phi^k(|v\rangle\langle v|)$. **Minimal output entropy** is for $v = |\omega\rangle$:

**Theorem (Lieb-Solovej ’12)**

For all $J$, all $K = J + k/2$, all concave $f : [0, 1] \to \mathbb{R}$, all $v \in \mathcal{H}_J$

$$\text{Tr}_K f \left( \Phi^k(v) \right) \geq \text{Tr}_K f \left( \Phi^k(|\omega\rangle) \right)$$

The Main Theorem ($k = \infty$) follows from the classical limit:

$$\lim_{k \to \infty} \frac{2J + 1}{2K + 1} \text{Tr}_K f \left( \frac{2K + 1}{2J + 1} \Phi^k(v) \right) = \frac{2J + 1}{4\pi} \int_{S^2} f \left( \Phi^\infty_v (\omega') \right) d\omega'.$$
We introduce the **Bosonic annihilation** operators $a_1, a_2$ and their adjoints the **creation** operators $a_1^*, a_2^*$, where $1, 2$ numbers basis $e_1, e_2$ of $\mathbb{C}^2$:

$$a_1^*, a_2^*: \bigoplus_J \mathcal{H}_J \to \bigoplus_J \mathcal{H}_J, \quad a_i^* (\mathcal{H}_J) \subseteq \mathcal{H}_{J + \frac{1}{2}} = \bigotimes_{\text{sym}} \mathbb{C}^2$$

$$a_i^* \phi = \sqrt{2J + 1} P_{\text{sym}} (e_i \otimes \phi) \text{ for } \phi \in \mathcal{H}_J.$$

Then $\Phi^k(v) = \frac{(2J + 1)!}{(2K + 1)!} \sum_{i_1, \ldots, i_k} a_{i_1}^* \cdots a_{i_k}^* |v\rangle \langle v| a_{i_k} \cdots a_{i_1}$

Note $\Phi^k = \Phi^1 \circ \Phi^{k-1}$. Moreover, we introduce the dual map $\Phi^{k*}$ of operators on $\mathcal{H}_{J + k/2}$ to operators on $\mathcal{H}_J$, i.e.,

$$\text{Tr}_K (\rho' \Phi^k(\rho)) = \text{Tr}_K (\Phi^{k*}(\rho')(\rho))$$

The channels $\Phi^k$ are also known as the **Universal Quantum Cloning Machines**.
Formulation in terms of majorization

Alternatively to using traces of concave functions the previous theorem may be equivalently rephrased as

**Theorem**

For all $v \in \mathcal{H}_J$ and all $\omega \in \mathbb{S}^2$ the sequence of eigenvalues of $\Phi^k(|\omega\rangle)$ majorizes the sequence of eigenvalues of $\Phi^k(v)$.

**Def.** $a_1 \geq a_2 \geq \cdots \geq a_M$ majorizes $b_1 \geq b_2 \geq \cdots \geq b_M$ if

\[
\begin{align*}
    a_1 &\geq b_1, \\
    a_1 + a_2 &\geq b_1 + b_2, \\
                  \vdots \\
    a_1 + a_2 + \cdots + a_{M-1} &\geq b_1 + b_2 + \cdots + b_{M-1}, \\
    a_1 + a_2 + \cdots + a_M &= b_1 + b_2 + \cdots + b_M
\end{align*}
\]
**Inductive argument**

**Eigenvalues** (decreasing) and **eigenvectors** of $\Phi^k(v)$: $\lambda^k_j$, $\phi^k_j$.

$\Phi^k(v)$ has rank $k + 1$.

**Eigenvalues** of $\Phi^k(|\omega\rangle)$: $\lambda_j^{k,C}$. $|\omega\rangle$ is pure tensor product in $\mathcal{H}_J = \bigotimes_{\text{sym}}^{2J} \mathbb{C}^2$. Eigenvalues of $\Phi^k(|\omega\rangle)$ are explicit.

Will prove by **induction** on $m$ and $k$ that

$$\sum_{j=1}^{m} \lambda_j^k \leq \sum_{j=1}^{m} \lambda_j^{k,C} \quad (1)$$

**• $m = 1$:** $\lambda_1^k = \| \Phi^k(v) \| \leq \frac{2J+1}{2K+1} = \lambda_1^{k,C}$

**• $m \geq 2$ assume (1) OK up to $m - 1$:** Use induction on $k$:
  - (1) is identity for $k + 1 \leq m$ by trace preservation so OK for $k = 1$.
  - **Assume (1) OK for $k - 1$:** We write

$$\sum_{j=1}^{m} \lambda_j^k = \sum_{j=1}^{m} \langle \phi_j^k, \Phi^k(v) \phi_j^k \rangle$$
Using duality

\[
\sum_{j=1}^{m} \lambda_j^k = \text{Tr} \ K \left[ \sum_{j=1}^{m} |\phi_j^k\rangle\langle\phi_j^k| \Phi^k(v) \right]
\]

\[
= \text{Tr} \ K \left[ \Phi^1 \left( \sum_{j=1}^{m} |\phi_j^k\rangle\langle\phi_j^k| \right) \Phi^{k-1}(v) \right] = \text{Tr} \ K \left[ \Gamma \Phi^{k-1}(v) \right]
\]

\[
\Gamma = \Phi^1 \left( \sum_{j=1}^{m} |\phi_j^k\rangle\langle\phi_j^k| \right) = \frac{1}{(2K+1)} \sum_{j=1}^{m} \sum_{i} a_i |\phi_j^k\rangle\langle\phi_j^k| a_i^*
\]

**Two simple observations** about the operator \( \Gamma \):

\[0 \leq \Gamma \leq I, \quad \text{Tr} \ \Gamma = \frac{2K}{2K+1} m = m - 1 + \frac{2K+1-m}{2K+1}.
\]

Using **bathtub principle** we get **upper bound** if we **replace** \( \Gamma \) by

\[
\Gamma_{\text{optimal}} = \sum_{j=1}^{m-1} |\phi_j^{k-1}\rangle\langle\phi_j^{k-1}| + \frac{2K+1-m}{2K+1} |\phi_m^{k-1}\rangle\langle\phi_m^{k-1}|,
\]
A recursive inequality

Inserting

\[ \Gamma_{\text{optimal}} = \sum_{j=1}^{m-1} |\phi_j^{k-1}\rangle\langle\phi_j^{k-1}| + \frac{2K+1-m}{2K+1} |\phi_m^{k-1}\rangle\langle\phi_m^{k-1}|, \]

gives

\[ \sum_{j=1}^{m} \lambda_j^k = \text{Tr} \left( K^{-\frac{1}{2}} \left[ \Gamma \Phi^{k-1}(\nu) \right] \right) \]

\[ \leq \text{Tr} \left( K^{-\frac{1}{2}} \left[ \Gamma_{\text{optimal}} \Phi^{k-1}(\nu) \right] \right) \]

\[ \leq \sum_{j=1}^{m-1} \lambda_j^{k-1} + \frac{2K+1-m}{2K+1} \lambda_m^{k-1} \]

using induction:

\[ \leq \sum_{j=1}^{m-1} \lambda_j^{k-1,C} + \frac{2K+1-m}{2K+1} \lambda_m^{k-1,C} = \sum_{j=1}^{m} \lambda_j^{k,C} \]

Last identity easy to check for \( \lambda_j^{k,C} \). Induction complete!
The classical limit (only one sided inequality)

Will show a version of the **Berezin-Lieb inequality**: For $f$ concave

$$\frac{2J + 1}{2K + 1} \text{Tr}_K f \left( \frac{2K + 1}{2J + 1} \Phi^k(v) \right) \leq \frac{2J + 1}{4\pi} \int_{S^2} f \left( |\langle v|\omega\rangle_J|^2 \right)$$

If $v = |\omega_0\rangle$ right side explicitly limit $K \rightarrow \infty$ of left side:

**That is all we need!**

**Jensen’s inequality** gives above **Berezin-Lieb-inequality**:

$$\text{Tr}_K f \left( \frac{2K+1}{2J+1} \Phi^k(v) \right) = \sum_j f \left( \frac{2K+1}{2J+1} \lambda_j^k \right) \frac{2K+1}{4\pi} \int_{S^2} |\langle \phi_j^k|\omega\rangle_K|^2 d\omega$$

$$\leq \frac{2K+1}{4\pi} \int_{S^2} f \left( \sum_j \frac{2K+1}{2J+1} \lambda_j^k |\langle \phi_j^k|\omega\rangle_K|^2 \right) d\omega$$

$$= \frac{2K+1}{4\pi} \int_{S^2} f \left( \frac{2K+1}{2J+1} \langle \omega, \Phi^k(v)\omega\rangle_K \right) d\omega$$

$$= \frac{2K+1}{4\pi} \int_{S^2} f \left( |\langle v|\omega\rangle_J|^2 \right) d\omega.$$