Asymptotic completeness in quantum field theory with massless bosons

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based on work with A. Kupiainen, and partially on ongoing work with A. Kupiainen and M. Griesemer.

Main goal

Prove textbook properties of interacting quantum field theories

- Relaxation to the ground state at $T = 0$.
- Scattering Theory, Asymptotic completeness: After some time, the interacting field consists of freely evolving 'dressed' particles (Quasiparticles)

Models

- We always impose ultraviolet cutoff.
- Possible aim: 'Non-Relativistic Quantum Electrodynamics' (photon field, but electrons non-relativistic) $\Rightarrow$ Pauli-Fierz model.
- Include (infinitely heavy) atom $\Rightarrow$ break translation-invariance
- Atom+electron modelled by finite-level system (levels $\sim$ bound states of atom) $\Rightarrow$ paradigm of open Q systems

'Generalized' Spin-boson model

- Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{at}} \otimes \mathcal{H}_{\mathrm{F}}$
  $\mathcal{H}_{\mathrm{at}} = \mathbb{C}^m$ (atom space) and $\mathcal{H}_{\mathrm{F}} = \Gamma(L^2(\mathbb{R}^3))$ (photon field).

$$\mathcal{H}_{\mathrm{F}} \ni \Psi_{\mathrm{F}} = c \underbrace{\Omega}_{vac.} + \underbrace{\Psi_1(q)}_{1-phot.} + \underbrace{\Psi_1(q,q')}_{2-phot.} + \underbrace{\Psi_1(q,q',q'')}_{3-phot.} + \dots$$

  $q, q', q'' . \dots \in \mathbb{R}^3$ are the momenta of the photons
- Hamiltonian: $H_{\mathrm{at}} : \mathrm{diag}(E_1 . E_2, \dots, E_m)$ and weak coupling.

$$H = H_{\mathrm{at}} \otimes 1 + 1 \otimes H_{\mathrm{F}} + \lambda H_{\mathrm{atF}}, \qquad 0 < |\lambda| < 1$$

- Free photon Hamiltonian $H_{\mathrm{F}} = \int \mathrm{d}q |q| a_q^* a_q$, with $a_q^* . a_q$ creation/annihilation operators.

$$[a_q^*, a_{q'}] = -\delta(q - q')$$

$$a_q^* \, |k \text{ phot.}\rangle = |\text{same } k \text{ phot. + one with momentum } q\rangle$$

'Generalized' Spin-boson model

- Atom-photon coupling: $D = D^* \in \mathcal{H}_{\mathrm{at}}$

$$H_{\mathrm{atF}} = D \otimes \int \mathrm{d}q \, (\phi(q) \otimes a_q + \overline{\phi}(q) \otimes a_q^*)$$

with form factor $\phi(q) \sim |q|^{-1+\alpha/2+\epsilon}$ as $q \to 0$.

- If $\alpha > 0$, H bounded below. If $\alpha > 1$, H has ground state:

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- If $\alpha > 0$, H bounded below. If $\alpha > 1$, H has ground state: Check in Toy-model ($D = 1$)

$$H_q := |q| a_q^* a_q + \phi(q) a_q + \overline{\phi}(q) a_q^* = |q| b_q^* b_q - \frac{|\phi(q)|^2}{|q|}$$

with $b_q = a_q + \frac{\bar{\phi}}{|q|}$ (Bogoliubov trf.) Hence

$$H = \int^{\oplus} \mathrm{d}q H_q \geq \| \frac{\phi}{\sqrt{|q|}} \|_2^2, \qquad \text{in fact } \inf \sigma(H) = \| \frac{\phi}{\sqrt{|q|}} \|_2^2$$

GS energy finite $\quad \Leftrightarrow \quad \| \frac{\phi}{\sqrt{|q|}} \|_2^2 < \infty \quad \Leftarrow \quad \alpha > 0$.

'Generalized' Spin-boson model

$$H_q = |q| b_q^* b_q - \frac{|\phi(q)|^2}{|q|}$$

with $b_q = a_q + \frac{\bar{\phi}}{|q|} \Rightarrow$ ground state= 'twisted vacuum' Ω_b s.t.

$$b_q \Omega_b = (a_q + \frac{\bar{\phi}}{|q|}) \Omega_b = 0, \qquad \forall q$$

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$$\Rightarrow \qquad \langle \Omega_b, N \Omega_b \rangle = \int \mathrm{d}q \langle \Omega_b, a_q^* a_q \Omega_b \rangle = \| \frac{\phi}{|q|} \|_2^2$$

Hence
GS has finite #photons $\Leftrightarrow$ $\| \frac{\phi}{|q|} \|_2^2 < \infty$ $\Leftarrow$ $\alpha > 1$.

Careful analysis yields in fact: $\alpha > 1 \quad \Rightarrow \Omega_b \in \mathcal{H}_{\mathrm{F}}$.

Results I

- Fermi Golden Rule condition (explained later)
- Correlation decay (explained later)

$$\int \mathrm{d}t |\zeta(t)|(1+|t|)^{\alpha} < \infty, \qquad \zeta(t) := \int \mathrm{d}q |\phi(q)|^2 \mathrm{e}^{\mathrm{i}|q|t}$$

Write $\langle A \rangle_t = \mathrm{Tr}\, \rho \mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}$ with ρ density matrix and A observable.

Approach to steady state (DR, Kupiainen, arXiv:1109.5582)

Assume Fermi Golden Rule and Correlation decay with $\alpha > 0$, then

$$|\langle A \rangle_t - \langle A \rangle_\infty| \leq C(1+|t|)^{-\alpha}$$

for sufficiently local A and ρ. (Example, $\rho = \rho_{\mathrm{at}} \otimes |\Omega\rangle\langle\Omega|$ and $A = A_{\mathrm{at}} \otimes 1$). State $\langle A \rangle_\infty$ equals $\langle \Psi_{gs}, A\Psi_{gs} \rangle$ whenever Ψ_{gs} exists.

Results II

Consider approximate solution of EOM

$$\Phi_t(f_1, \dots, f_m) := a^*(\mathrm{e}^{-\mathrm{i}t|q|} f_1) \dots a^*(\mathrm{e}^{-\mathrm{i}t|q|} f_m) \mathrm{e}^{-\mathrm{i}tE_{gs}} \Psi_{gs}$$

with $a^*(f) = \int \mathrm{d}q f(q) a^*_q$. Easy to see:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\dots) - H\Phi_t(\dots) \to 0, \qquad t \to \infty$$

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Asymptotic completeness (in progress)

Take $\alpha > 1$ and $|\lambda| \ll 1$. In particular, $\exists ! \Psi_{gs}$. As $t\to\infty$, any solution of *EOM* Ψ_t is a linear superposition of different $\Phi_t(\dots)$.

Earlier work (In all cases: asymptotic completeness $\Rightarrow$ Relaxation)

- Exactly solvable models (atom potential harmonic) (Arai)
- Small perturbation thereof (Maassen, Spohn)
- Sharp Infrared cutoff (e.g. massive photons): (Derezinski, Gerard, Frohlich, Griesemer, Schlein, 2005)

Some easier questions: Markovian limits

Let $\rho_{\mathrm{at},t} = \mathrm{Tr}_{\mathrm{F}}\, \mathrm{e}^{-\mathrm{i}tH} \rho \mathrm{e}^{\mathrm{i}tH}$ and assume all eigenvalues of H_{at} simple.

General Idea (Van Hove '50, Davies '74)

$$\lim_{t=\lambda^{-2}\tau, \lambda \to 0} \rho_{\mathrm{at},t} = \mathrm{e}^{\tau \mathcal{L}} \rho_{\mathrm{at},0}$$

with $\mathcal{L}$ a Lindbladian (Quantum Markov generator). Limit corresponds to $t_{\mathrm{Fcorrelations}} \ll t_{dissipation} \sim \lambda^{-2}$ (phonons are like white noise).

Write $\rho_{\mathrm{at},0} = \mathrm{diag}(\mu_1(0), \ldots, \mu_m(0)) + \rho_{\text{off-diag},0}$, then

$$\mathrm{e}^{\tau \mathcal{L}} \rho_{\mathrm{at},0} = \mathrm{diag}(\mu_1(\tau), \ldots, \mu_m(\tau)) + \rho_{\text{off-diag},\tau},$$

where $\overline{\mu}(\tau)$ is a jump process on $\sigma(H_{\mathrm{at}})$. Jumps $e \to e'$, $\sim$ emission of photon with $|q| = e - e'$.

Furthermore, Decoherence: $\|\rho_{\text{off-diag},\tau}\| \sim \mathrm{e}^{-c\tau}$

Character of jump process

- The jump rate $j(e \to e')$ is calculated from second-order perturbation theory:

$$j(e \to e') = 2\pi |\langle e|D|e'\rangle|^2 \int \mathrm{d}q \delta(e - e' - |q|) |\phi(q)|^2$$

- Iff directed graph with edges $(e \sim e') \Leftrightarrow j(e \to e') \neq 0$ is connected, then the jump process converges to the state $e_{gs} = \min\{e \in \mathrm{sp} H_{\mathrm{at}}\}$ exponentially fast (Perron-Frobenius theorem): $\Rightarrow$ atom cascades down to ground state. This (together with uniqueness of e_{gs}) is our **Fermi Golden Rule condition**.

Non-Markovian corrections

Since the photon field is not white noise, the true evolution is not Markovian:

Correlation function $\zeta(t) := \langle \Omega, \Phi(t)\Phi(0)\Omega \rangle = \int \mathrm{d}q |\phi(q)|^2 \mathrm{e}^{\mathrm{i}|q|t}$

where $\Phi(t) := \int \mathrm{d}q \phi(q) \mathrm{e}^{\mathrm{i}t|q|} a_q + h.c.$.

- Unless ϕ has sharp infrared cutoff, $\zeta(t)$ cannot decay exponentially (contrast with Huygens principle for the wave equation.) The decay is $\zeta(t) = O(t^{-(1+\alpha)})$.
- In general, one should not expect time-correlation of observables to decay faster than $\zeta(t)$. However, in the Markovian approximation, atom observables decorrelate exponentially. (Long-standing confusion in physics literature: Adler-Wainwright, Slow decorrelation in gases causes anomalous diffusion in $d = 1, 2$)

Strategy of proof

- We know that on time scales $t \approx \lambda^{-2}$, the atom evolves according to a finite state Markov jump process.
- The corrections to this behaviour are manifestly non-Markovian and long-range in time.
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- We know that on time scales $t \approx \lambda^{-2}$, the atom evolves according to a finite state Markov jump process.
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- This looks like doing perturbation theory around a stochastic system, rather than around the unperturbed Hamiltonian system. $\Rightarrow$ easier (because behaviour stable under perturbation)
- Previous results used spectral perturbation theory: complex spectral deformations, resonance theory. Energy-domain expansion.
- Our Time-domain expansion appears simpler and more unified (positive temp, non-equilibrium setup), but seems too clumsy to deal with ionization threshold: Duhamel expansion always excites ionizes electron.

Polymer models: Problem

1D Polymer model

Let $A_i \subset [1,N] = \{1,2,\dots,N\}$ and consider partition function

$$Z_N := \sum_{(A_1,\dots,A_k), A_i \nsim A_j} \prod_{j=1}^{k} w(A_j)$$

for some polymer weights $w(A) \in \mathbb{C}$ and adjacency relation $\sim$ (think: $A \sim A' \Leftrightarrow A \cap A' \neq \emptyset$).

Goal:

Map it into stat-mech: Find 'local free energy functional' $A \mapsto w^T(A) \in \mathbb{C}$ such that

$$\log Z = \sum_{A \subset [1,N]} w^T(A)$$

Polymer models: Solution

Assume the 'Kotecky-Preiss' criterion:

$$\sum_{A \sim A'} \mathrm{e}^{a|A|} w(A) \leq a|A'|, \qquad \text{for some } a > 0.$$

Then $\exists$ local weights w^T:

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Example of use: Estimate free energy density $f := \lim_N N^{-1} \log Z_N$

$$f = \frac{1}{N} \sum_{A\subset[1,N]} w^T(A) \sim \sum_{A, \min A=0} w^T(A) + o(1), \qquad N \to \infty$$

since the sum is absolutely convergent.

By Vitali theorem, if $w(A)$ depends analytically on a parameter and KP condition holds uniformly, then f is also analytic.

Summary

Cluster expansion for polymer models

Key steps were

- independence of polymers $Z = \sum_{\dots} \prod_i w(A_i)$ with $w(A_i)$ numbers.
- Summability of polymer weights , because (KP criterion)

$$\sum_{A \sim A'} \mathrm{e}^{a|A|} w(A) \leq a|A'|$$

- Smallness of polymer weights: KP criterion requires to take a larger as $w(A)$ grows.

We will calculate by analogous methods

$$Z = \operatorname{Tr} \rho_t = 1, \qquad \rho_0 = \rho_{\mathrm{at},0} \otimes \rho_{\mathrm{F}}$$

with ρ_{F} vacuum state. If expansion sufficiently robust $\Rightarrow$ easy to modify so as to get $\operatorname{Tr} O \rho_t$. Note: Our problem not given as partition function.

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Introduce discrete structure

$$Z = \text{Tr}\,\rho_t = \text{Tr}(U^N \rho_0)$$

where $t = \lambda^{-2} N$ and

$$U : \mathscr{B}_1 \to \mathscr{B}_1 : \;\; U\rho = \mathrm{e}^{-\mathrm{i}\lambda^{-2}H} \rho \mathrm{e}^{\mathrm{i}\lambda^{-2}H}$$

Far from our needs: We have a product of operators, instead of numbers.

To bring in smallness, split

$$U = T \otimes U_{\mathrm{F}} + B$$

where U_{F} is free F-dynamics and

$$T : \mathcal{B}_1(\mathcal{H}_{\mathrm{at}}) \to \mathcal{B}_1(\mathcal{H}_{\mathrm{at}}) : \qquad T\rho_{\mathrm{at}} = \mathrm{Tr}_{\mathrm{F}} \left[U(\rho_{\mathrm{at},0} \otimes \rho_{\mathrm{F}}) \right],$$

with $\rho_{\mathrm{F}} = |\Omega\rangle\langle\Omega|$.
B small because we are close to the Markovian limit (recall time $\sim \lambda^{-2}$ in U)

$$T = \mathrm{e}^{\mathcal{L}} + o(1), \qquad \lambda \to 0$$

Hence, by simple perturbation theory, the operator T is ergodic:

$$T^n \quad \to_{n \to \infty} \quad P = |\rho_{inv}\rangle\langle 1|$$

for some ρ_{inv}. (Because $\mathrm{e}^{\tau\mathcal{L}}$ is ergodic by assumption)
Bold Approximation: Replace T by P.

Hence, by $T \to P$.

$$Z = \mathrm{Tr}(U^N \rho_0) = \mathrm{Tr}((T \otimes U_{\mathrm{F}} + B)^N \rho_0) \sim \mathrm{Tr}((P \otimes U_{\mathrm{F}} + B)^N \rho_0).$$

Number P, U_{F}, B so that we can write ($\mathcal{T}$ is time-ordering)

$$Z = \mathrm{Tr}\, \mathcal{T} \prod_{i=1}^{N} (P_i \otimes U_{\mathrm{F},i} + B_i)\rho_0.$$

Go to interaction picture w.r.t free F-dynamics:

$$B_i^I := (U_{\mathrm{F}})^{-i} B_i (U_{\mathrm{F}})^{i-1}$$

and, writing $P \otimes 1 \to P$ and $B_i^I \to B_i$, and using $U_{\mathrm{F}} \rho_0 = \rho_0$,

$$Z = \mathrm{Tr}\, \mathcal{T} \prod_{i=1}^{N} (P_i + B_i)\rho_0).$$

Now there is a genuine time-dependence in B_i.

$$Z = \operatorname{Tr} \mathcal{T} \prod_{i=1}^{N} (P_i + B_i)\rho_0) = \sum_{A \subset [1,N]} \operatorname{Tr} \mathcal{T} \left(\prod_{i \in A^c} P_i \prod_{i \in A} B_i \right) \rho_0$$

Where is the Independence? How to write it as a product of numbers?

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Where is the Independence? How to write it as a product of numbers? Assume that B_i are of the form $B_{\mathrm{at},i} \otimes 1$.
Then it factorizes (up to boundary effect) because

$$\operatorname{Tr} \left(\mathcal{T} \prod_{i \in A^c} P_i \prod_{i \in A} B_{\mathrm{at},i} \right) P \rho_0 = \prod_j \operatorname{Tr}_{\mathrm{at}} P \left(\mathcal{T} \prod_{i \in A_j} B_{\mathrm{at},i} \right) P$$

where $A = \cup_j A_j$ decomposition into non-adjacent intervals, and we used $\operatorname{Tr} P \rho_0 = 1$.
$\Rightarrow$ We did obtain

$$Z = \sum_{\dots} \prod_j w(A_j)$$

Back to

$$Z = \operatorname{Tr} \mathcal{T} \prod_{i=1}^{N} (P_i + B_i) \rho_0).$$

Now assume instead that $B_j = 1 \otimes B_{\mathrm{F},j}$ with

$$B_{\mathrm{F},j} = \lambda \Phi(\lambda^{-2} j), \qquad \Phi(t) = (a(\mathrm{e}^{-\mathrm{i}t\omega}\phi) + h.c.).$$

(Recall that $\lambda^{-2} j$ is the real time corresponding to block j) Then P_i can be dropped and Wick theorem gives factorization:

$$\begin{aligned} \operatorname{Tr} \left(\mathcal{T} \prod_{j \in A} B_{\mathrm{F},j} \right) \rho_0 &= \prod_{\pi \in \mathrm{Pair}(A)} \prod_{(j,j') \in \pi} \operatorname{Tr}(B_{\mathrm{F},j} B_{\mathrm{F},j'}) \rho_{\mathrm{F}} \\ &= \prod_{\pi \in \mathrm{Pair}(A)} \prod_{(j,j') \in \pi} \lambda^2 \zeta(\lambda^{-2} |j - j'|) \end{aligned}$$

$\Rightarrow$ Again, we did obtain

$$Z = \sum_{\dots} \prod_j w(A_j)$$

In reality, closer to (but still neglecting B_{at})

$$B_{\mathrm{F},j} = \sum_m \lambda^m \int_{\frac{j}{\lambda^2} \leq t_1 \leq \ldots \leq t_m \leq \frac{j+1}{\lambda^2}} \mathrm{d}t_1 \ldots dt_m \Phi(t_m) \ldots \Phi(t_1)$$

Hence instead of pairings between $B_{\mathrm{F},j}, B_{\mathrm{F},j'}$, arbitrary number of links (from pairings of Φ). Instead of factorizing A as a sum over pairings, we should factorize it as a sum over partitions.

$$\mathrm{Tr} \left(\mathcal{T} \prod_{j \in A} B_{\mathrm{F},j} \right) \rho_0 \quad = \sum_{(A_1,\ldots,A_k), \quad A_i \cap A_j = \emptyset, \quad \cup A_j = A} \prod_{j=1}^{k} w(A_j)$$

Then we get indeed

$$Z = \sum_{(A_1,\ldots,A_k),A_i \cap A_j = \emptyset} \prod_{j=1}^{k} w(A_j)$$

Where is summability ?
If $A = \{\tau_1, \tau_2, \ldots, \tau_m\}$ non-consecutive times, then $w(A)$ contains for example

$$\prod_{j=1,\ldots N} \lambda^2 \zeta(\lambda^{-2}|n_j - n_{j-1}|)$$

for some sequence $\overline{n} = (n_1, n_2, \ldots, n_N)$ with values in A and covering A, i.e. $\mathrm{Supp}\overline{n} = A$. More precisely

$$w(A) \sim \sum_{\overline{n}, \mathrm{Supp}\overline{n} = A} \prod_{j=1,\ldots N} \lambda^2 \zeta(\lambda^{-2}|n_j - n_{j-1}|)$$

Where is summability ? Hence we have

$$w(A) \sim \sum_{\overline{n}, \mathrm{Supp}\overline{n}=A} \prod_{j=1,\dots N} \lambda^2 \zeta(\lambda^{-2}|n_j - n_{j-1}|)$$

Now

$$\sum_{A, \min A = \tau_1} w(A) \sim \sum_{\overline{n}, n_1=\tau_1} \prod_{j=1,\dots N} \lambda^2 \zeta(\lambda^{-2}|n_j - n_{j-1}|)$$

$$\sim \sum_N \lambda^{2(N-1)} \dots \sum_{n_{N-1}} \zeta(\lambda^{-2}|n_{N-1} - n_{N-2}|) \sum_{n_N} \zeta(\lambda^{-2}|n_N - n_{N-1}|$$

$$\sim \sum_N \lambda^{2(N-1)} \|\zeta\|_1^{N-1} \sim \lambda^2 \|\zeta\|_1$$

smallness also follows. Hence, KP criterion holds and we can exponentiate the expansion.

Application: convergence to stationary state

The KP criterion allows to exponentiate the representation,

$$\log Z = \sum_{A \subset [1,N]} w^T(A), \qquad \text{truncated weights } w^T(\cdot)$$

with bounds ($a \sim |\lambda|^{2\alpha}$ emerges as small parameter)

$$\sum_{A : A \sim A_0} \left| w^T(A) \right| \leq a |A_0|$$

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Why useful?

Dependence on initial state $\rho_{\mathrm{at},0}$ is only in $w^T(A), A \ni 1$

Do $Z = \mathrm{Tr}\, O \rho_t$, then dependence on observable O is only in $w^T(A), A \ni N$

Since dependence on observable O is only in $w^T(A), A \ni N$

$$\log \frac{Z(\rho_{\mathrm{at},0}, O)}{Z(\rho_{\mathrm{at},0}, 1)} = \sum_{A \ni N} \Big(w^T(A, \rho_{\mathrm{at},0}, O) - w^T(A, \rho_{\mathrm{at},0}, 1) \Big)$$

$$\log \frac{Z(\rho'_{\mathrm{at},0}, O)}{Z(\rho'_{\mathrm{at},0}, 1)} = \sum_{A \ni N} \Big(w^T(A, \rho'_{\mathrm{at},0}, O) - w^T(A, \rho'_{\mathrm{at},0}, 1) \Big)$$

But, since dependence on initial state is only in $w^T(A), A \ni 1$, we can divide to get

$$\log \frac{Z(\rho_{\mathrm{at},0}, O)}{Z(\rho'_{\mathrm{at},0}, O)} = \sum_{\{1,N\} \subset A} \big(v(A, \rho_{\mathrm{at}}, O) - v(A, \rho'_{\mathrm{at}}, O) \big)$$

where $v(A, \rho_{\mathrm{at}}, O) = w^T(A, \rho_{\mathrm{at},0}, O) - w^T(A, \rho_{\mathrm{at},0}, 1)$.

Conclusion

We have obtained.

$$\log \frac{Z(\rho_{\mathrm{at},0}, O)}{Z(\rho'_{\mathrm{at},0}, O)} = \sum_{\{1,N\}\subset A} \left(v(A, \rho_{\mathrm{at}}, O) - v(A, \rho'_{\mathrm{at}}, O) \right)$$

where $v(A, \rho_{\mathrm{at}}, O) = w^T(A, \rho_{\mathrm{at},0}, O) - w^T(A, \rho_{\mathrm{at},0}, 1)$.

Of course, $w^T(A)$ with $\{1, N\} \subset A$ decay as $N \to \infty$. (Contain very long pairings). This allows to establish that

$$\frac{\mathrm{Tr}(\rho_t O)}{\mathrm{Tr}(\rho'_t O)} = \frac{Z(\rho_{\mathrm{at},0}, O)}{Z(\rho'_{\mathrm{at},0}, O)} \to 1$$

where ρ_t, ρ'_t are the time-evolutions of $\rho_{\mathrm{at}} \otimes \rho_{\mathrm{F}}, \rho'_{\mathrm{at}} \otimes \rho_{\mathrm{F}}$, respectively.

Hence asymptotic state does not depend on initial state.