Lieb-Thirring inequalities for anyons

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joint work with
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Outline of Talk

1. Identical particles and statistics
2. Fermions vs. bosons vs. anyons?
3. Lieb-Thirring inequalities for anyons
4. Hardy inequalities for anyons
5. A local exclusion principle for anyons

Identical particles and statistics

M. Leinaas, J. Myrheim, *On the Theory of Identical Particles*, Il Nuovo Cimento 37B, 1977

Configuration space for N identical particles:

$$X_d^N := \left(\mathbb{R}^{dN} \setminus \mathbb{D}\right) \Big/ S_N,$$

$$\mathbb{D} := \{(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N) = x \in \mathbb{R}^{dN} : \boldsymbol{x}_j = \boldsymbol{x}_k, j \neq k\}$$

$d \geq 3$**:**

$\pi_1(X_d^N) = S_N$, the permutation group

One-dimensional unitary representations $\rho : S_N \to U(1)$:

- $\rho(\sigma_{jk}) = +1$: **bosons** (e.g. photons), $u \in \bigotimes_{\mathsf{sym}}^N L^2(\mathbb{R}^d)$
- $\rho(\sigma_{jk}) = -1$: **fermions** (e.g. electrons), $u \in \bigwedge^N L^2(\mathbb{R}^d)$

$$u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_N) = \pm u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_N)$$

Identical particles and statistics

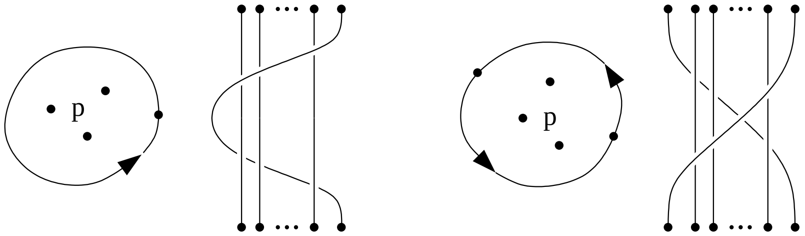
$d = 2$**:**

$\pi_1(X_2^N) = B_N$, the braid group on N strands

One-dimensional unitary representations $\rho : B_N \to U(1)$ characterized by a single phase $\rho(\sigma_{jk}) = e^{i\alpha\pi}$

- $\alpha = 0$: bosons
- $\alpha = 1$: fermions
- **any** phase/statistics parameter $\alpha \in [0, 2)$: **anyons**

Identical particles and statistics



exp(2p alpha i) exp((2p+1)alpha i)

One- resp. two-particle interchange loops with corresponding braid diagrams (where we can think of time as running upwards) and phases.

Modelling anyons

- **Anyon gauge**: square-integrable sections of a locally flat complex line bundle over X_2^N ("multivalued" wavefunctions):

$$u \in \Gamma(L), \quad L \to X_2^N$$

$$\text{momenta } \nabla_j u$$

- **Magnetic gauge**: bosons on $\mathbb{R}^2$ with Aharonov-Bohm magnetic potentials attached to each particle:

$$u \in \mathcal{D}_N := C_0^\infty(\mathbb{R}^{2N} \setminus \mathbb{D}) \cap \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^2)$$

$$D_j u := (-i\nabla_j + \boldsymbol{A}_j(\boldsymbol{x}_j))\, u, \quad \boldsymbol{A}_j(\boldsymbol{x}) := \alpha \sum_{k \neq j} (\boldsymbol{x} - \boldsymbol{x}_k)^{-1} I$$

Fermions

EH Lieb, WE Thirring, Bound for the kinetic energy of fermions which proves the stability of matter, PRL **35** (1975)

Pauli's exclusion principle: $u \in \bigwedge^N L^2(\mathbb{R}^d)$ normalized $\Rightarrow$
Lieb-Thirring inequality :

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} \left(|\nabla_j u|^2 + V(\boldsymbol{x}_j)|u|^2\right) dx \;\geq\; -C_d^{\mathrm{LT}} \int_{\mathbb{R}^d} |V_-(\boldsymbol{x})|^{1+\frac{d}{2}}\, d\boldsymbol{x},$$

$\Leftrightarrow$ Kinetic energy inequality:

$$T := \sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 \, dx \;\geq\; C_d^{\mathrm{K}} \int_{\mathbb{R}^d} \rho(\boldsymbol{x})^{1+\frac{2}{d}}\, d\boldsymbol{x},$$

$$\rho(\boldsymbol{x}) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |u(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j = \boldsymbol{x}, \ldots, \boldsymbol{x}_N)|^2 \prod_{k \neq j} d\boldsymbol{x}_k$$

Bosons and anyons

For **bosons** (or in general $u \in L^2(\mathbb{R}^{dN})$ only the uncertainty principle:

$$T = \sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_j u|^2 \, dx \;\geq\; \frac{C_d^{\mathrm{K}}}{N^{2/d}} \int_{\mathbb{R}^d} \rho(\boldsymbol{x})^{1+\frac{2}{d}} \, d\boldsymbol{x},$$

What about $0 < \alpha < 1$ **?**

$$T := \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j u|^2 \, dx, \quad D_j := -i\nabla_j + \alpha \sum_{k \neq j} (\boldsymbol{x}_j - \boldsymbol{x}_k)^{-1} I$$

We prove local Hardy inequality for anyons

$\Rightarrow$ local exclusion principle, strength depending on α

$\Rightarrow$ Lieb-Thirring inequality for certain α

Lieb-Thirring inequalities for anyons

Theorem (Kinetic energy inequality for anyons)

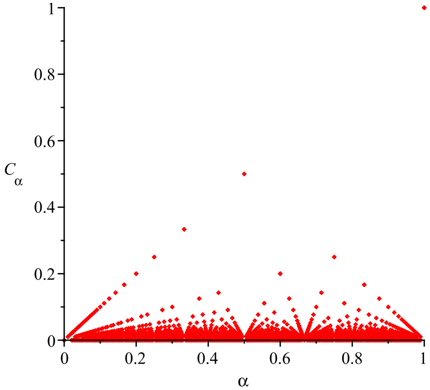
Let $u \in \mathcal{D}_N$ *Then*

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j u|^2 \, dx \; \geq \; C_{\mathrm{K}} C_\alpha^2 \int_{\mathbb{R}^2} \rho(\boldsymbol{x})^2 \, d\boldsymbol{x},$$

for some positive constant C_{K}.

$$\begin{aligned} C_\alpha \;\; &:= \;\; \inf_{N \in \mathbb{N}} C_{\alpha,N} = \inf_{p,q \in \mathbb{Z}} |(2p+1)\alpha - 2q| \\ &= \;\; \begin{cases} \frac{1}{\nu}, & \text{if } \alpha = \frac{\mu}{\nu} \text{ is a reduced fraction with } \mu \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The statistics-dependent constant C_α



Hardy inequalities for anyons

Theorem (Many-anyon Hardy inequality)

$u \in \mathcal{D}_N$, $\Omega \subset \mathbb{R}^2$ *open convex. Then*

$$\int_{\Omega^N} \sum_{j=1}^{N} |D_j u|^2 \, dx \;\geq\; \frac{4C_{\alpha,N}^2}{N} \int_{\Omega^N} \sum_{i<j} \frac{|u|^2}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^2} \chi_{\Omega\circ\Omega}(\boldsymbol{x}_i, \boldsymbol{x}_j) \, dx,$$

with

$$\Omega \circ \Omega := \left\{ (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \Omega^2 : \tfrac{1}{2}|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \operatorname{dist}\left(\tfrac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2), \Omega^c \right) \right\},$$

and

$$C_{\alpha,N} := \min_{p \in \{0,1,\ldots,N-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|.$$

(cp. M. Hoffmann-Ostenhof, T. Hoffman-Ostenhof, A. Laptev, J. Tidblom, 2008)

Ideas of the proof

- Pairwise relative parameterization of Ω

$$\int_{\Omega^N} \sum_{j=1}^{N} |D_j u|^2 \, dx \geq \frac{1}{N} \sum_{j<k} \int_{\Omega^{N-2}} \int_{\Omega^2} |(D_j - D_k)u|^2 \, d\boldsymbol{x}_j d\boldsymbol{x}_k \, d\tilde{x}$$

$$\boldsymbol{R} := \frac{1}{2}(\boldsymbol{x}_j + \boldsymbol{x}_k), \quad \boldsymbol{r} := \frac{1}{2}(\boldsymbol{x}_j - \boldsymbol{x}_k)$$

$$v(\boldsymbol{R}; \boldsymbol{r}) := u(\boldsymbol{R} + \boldsymbol{r}, \boldsymbol{R} - \boldsymbol{r})$$

$$(D_j - D_k)u = \left(-i\nabla_{\boldsymbol{r}} + \alpha \boldsymbol{r}^{-1} I + \boldsymbol{A}_{jk}(\boldsymbol{R} + \boldsymbol{r}) - \boldsymbol{A}_{jk}(\boldsymbol{R} - \boldsymbol{r})\right) v$$

- Local magnetic Hardy inequality with symmetry:

Ideas of the proof

Lemma (Magnetic Hardy inequality with symmetry)

$\Omega = B_{R_2}(0) \setminus \bar{B}_{R_1}(0)$, $R_2 > R_1 \geq 0$ annular domain in $\mathbb{R}^2$.
Vector potential $\boldsymbol{a} : \Omega \to \mathbb{R}^2$, s.t.

- *$\nabla \wedge \boldsymbol{a} = 0$ on Ω*
- *$\int_\Gamma \boldsymbol{a} \cdot d\boldsymbol{r} = \Phi$ loop Γ in Ω enclosing $\bar{B}_{R_1}(0)$.*
- *$\boldsymbol{a}(-\boldsymbol{r}) = -\boldsymbol{a}(\boldsymbol{r})$, $\boldsymbol{r} \in \Omega$*
- *$v \in C^\infty(\Omega)$, $v(-\boldsymbol{r}) = v(\boldsymbol{r})$ for all $\boldsymbol{r} \in \Omega$. Then*

$$\int_\Omega |D_{\boldsymbol{r}} v|^2 \, d\boldsymbol{r} \;\geq\; \min_{q \in \mathbb{Z}} \left| \frac{\Phi}{2\pi} - 2q \right|^2 \int_\Omega \frac{|v|^2}{|\boldsymbol{r}|^2} \, d\boldsymbol{r},$$

where $D_{\boldsymbol{r}} := -i\nabla_{\boldsymbol{r}} + \boldsymbol{a}(\boldsymbol{r})$.

(cp. Laptev-Weidl 1998, Balinsky 2003, . . .)

A local exclusion principle for anyons

Lemma (Local energy / $\sim$ Pauli principle)

$u \in \mathcal{D}_N$, $\Omega \subseteq \mathbb{R}^2$ either disk or square, with area $|\Omega|$. Then

$$\int_{\Omega^N} \sum_{j=1}^{N} |D_j u|^2 \, dx \;\geq\; (N-1)\frac{c_\Omega C_{\alpha,N}^2}{|\Omega|} \int_{\Omega^N} |u|^2 \, dx, \qquad (1)$$

where c_Ω is a constant which satisfies $c_\Omega \geq 0.477$ for the disk, and $c_\Omega \geq 0.358$ for the square.

(cp. F. J. Dyson, A. Lenard, *Stability of Matter. I*, JMP 1967)

Idea of proof: use part of the kinetic energy together with $|D_j u| \geq |\nabla_j |u||$, and then bound the corresp. pairwise bosonic Neumann operator $H := -\Delta_{\boldsymbol{x}_j} - \Delta_{\boldsymbol{x}_k} + C_{\alpha,n}^2 |\boldsymbol{r}|^{-2} \chi_{\Omega \circ \Omega}$ on Ω^2

Ingredients of the proof

Lemma (Local Pauli principle)

$u \in \mathscr{D}_N$, $\Omega \subseteq \mathbb{R}^2$ either disk or square. Then

$$T_\Omega := \sum_{j=1}^{N} \int_{\mathbb{R}^{2N}} |D_j u|^2 \chi_\Omega(\boldsymbol{x}_j)\, dx \ \geq \ \frac{c_\Omega C_\alpha^2}{|\Omega|} \left(\int_\Omega \rho \quad - 1 \right).$$

Lemma (Local uncertainty principle)

Let $u \in \mathscr{D}_N$, $Q \subset \mathbb{R}^2$ a square with area $|Q|$. Then

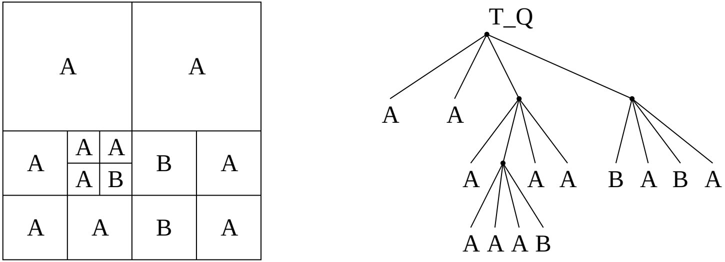
$$T_Q \ \geq \ c_1' \int_Q \rho^2 - C_2' \frac{(\int_Q \rho)^2}{|Q|}$$

Splitting algorithm

Start with a square $Q = Q_0 \subseteq \mathbb{R}^2$ s.t. $\operatorname{supp} u \subseteq Q_0$

- split $Q \to \bigcup_{j=1}^{4} Q'_j$ s.t. $|Q'_j| = |Q|/4$
- if $\int_{Q'_j} \rho < 2$: don't split Q'_j further, mark it A
- if all Q'_j marked A: back up to Q, mark it B (discard Q'_js)
- iterate splitting algorithm for each unmarked $Q'_j =: Q$

Splitting algorithm



Example of a splitting of Q_0 and a corresponding tree $\mathbb{T}$ of subsquares. For the B-square at level 3 in the tree, the set $\mathcal{A}(Q)$ of associated A-squares consists of 8 elements, while for the two B-squares at level 2, $\mathcal{A}(Q)$ coincide and has 4 elements.

Proof

Three classes of subsquares:

$\mathcal{A}_{\mathrm{c}}$: A-squares $Q \in \mathbb{T}$ s.t. $\displaystyle\int_Q \rho^2 \leq c \frac{(\int_Q \rho)^2}{|Q|}$, $(0 \leq \int_Q \rho < 2)$

$\mathcal{A}_{\mathrm{nc}}$: A-squares $Q \in \mathbb{T}$ s.t. $\displaystyle\int_Q \rho^2 > c \frac{(\int_Q \rho)^2}{|Q|}$, $(0 \leq \int_Q \rho < 2)$

$\mathcal{B}$: B-squares $Q \in \mathbb{T}$, $(2 \leq \int_Q \rho < 8)$

Local energy bounds:

$\mathcal{B}$**:** local Pauli energy + local uncertainty principle $\Rightarrow$

$$T_{Q_B} \geq C_\alpha^2 \left(c_1 \int_{Q_B} \rho^2 + c_2 \frac{(\int_{Q_B} \rho)^2}{|Q_B|} \right)$$

Proof cont.

$\mathcal{A}_{\rm nc}$**:** local uncertainty principle, for appropriate $c \Rightarrow$

$$T_{Q_A} \geq \frac{c_2'}{8} \int_{Q_A} \rho^2$$

$\mathcal{A}_{\rm c}$**:** negligible contribution to the energy compared to $\mathcal{B}$!
For each B-square Q_B (say at level k in $\mathbb{T}$):

$$\frac{(\int_{Q_B} \rho)^2}{|Q_B|} \geq \frac{4}{4^{-k}|Q_0|},$$

while for all A$_{\rm c}$-squares associated with Q_B:

$$\sum_{Q \in \mathcal{A}_{\rm c}(Q_B)} \int_Q \rho^2 \leq \sum_{j=1}^{k} \sum_{\substack{Q \in \mathcal{A}_{\rm c}(Q_B) \\ \text{at level } j}} c \frac{(\int_Q \rho)^2}{|Q|} \leq \sum_{j=1}^{k} 3c \frac{4}{4^{-j}|Q_0|} \leq 4c \frac{4^{k+1}}{|Q_0|}.$$

Proof cont.

Hence

$$T_{Q_B} \geq C_\alpha^2 \left(c_1 \int_{Q_B} \rho^2 + \frac{c_2}{4c} \sum_{Q \in \mathcal{A}_{\mathrm{c}}(Q_B)} \int_Q \rho^2 \right),$$

and

$$\begin{aligned} T &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j u|^2 \left(\sum_{Q_A \in \mathcal{A}} \chi_{Q_A}(\boldsymbol{x}_j) + \sum_{Q_B \in \mathcal{B}} \chi_{Q_B}(\boldsymbol{x}_j) \right) dx \\ &\geq \sum_{Q_A \in \mathcal{A}_{\mathrm{nc}}} T_{Q_A} + \sum_{Q_B \in \mathcal{B}} T_{Q_B} \ \geq \ C_{\mathrm{K}} C_\alpha^2 \int_{Q_0} \rho^2. \end{aligned}$$

□