Infinite systems of interacting chains with memory of variable length - a stochastic model for biological neural nets

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The model

- Huge system with $N \approx 10^{11}$ neurons that interact.
- Spike train : for each neuron i we indicate if there is a spike or not at time t.

$$X_t(i) \in \{0,1\}, X_t(i) = 1 \Leftrightarrow \text{ neuron } i \text{ has a spike at time } t \ .$$

- Actually, t is an index of the time window in which we observe the neuron. In the data we considered, the width of this window is typically 3 ms.

Background

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• Then : reset to a resting potential.

• Hence : **Variable length memory :** the memory of the neuron goes back up to its last spike – at least at a first glance.

• This is the framework considered by Cessac (2011) - but only for a **FINITE** number of neurons.

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Chain

$$X_t = (X_t(i), i \in \mathcal{I}), X_t(i) \in \{0,1\}, t \in \mathbb{Z},$$

$\mathcal{I}$ countable is the set of neurons. **We will work in the case where $\mathcal{I}$ is infinite**.

Time evolution : At each time step, neurons update independently from each other.

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Time evolution : At each time step, neurons update independently from each other.
This means : For any finite subset J of neurons,

$$P(X_t(i) = a_i, i \in J | \mathcal{F}_{t-1}) = \prod_{i \in J} P(X_t(i) = a_i | \mathcal{F}_{t-1}),$$

where

$$\mathcal{F}_{t-1} \text{ is the past history up to time } t-1 \ .$$

The model II

$$P(X_t(i) = 1 | \mathcal{F}_{t-1}) = \Phi\left(\sum_j W_{j \to i} \sum_{s=L_t^i}^{t-1} g(t-s) X_s(j), t - L_t^i\right).$$

Here :

- $W_{j \to i}$: **synaptic weight** of neuron j on i.
- $L_t^i = \sup\{s < t : X_s(i) = 1\}$ last spike strictly before time t in neuron i.
- $g : \mathbb{N} \to \mathbb{R}_+$ describes an aging effect. If there is no aging, then $g \equiv 1$.

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Either excitatory : $W_{j \to i} > 0$.
Or inhibitory : $W_{j \to i} < 0$.

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So it is an interesting mathematical object....

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This is both a mathematical and a biological question,

and there are experimental facts that we have to explain. . .

Next three slides slightly more technical

The proof of existence and uniqueness is based on the study of the transition probability

$$(1) \quad p_{(i,t)}(1|x) = \Phi\left(\sum_{j \neq i} W_{j \to i} \sum_{s = L_t^i(x)}^{t-1} g(t-s) x_s(j), t - L_t^i(x) \right) :$$

which depends on the space-time configuration of spike times

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For the specialists : The function $x \mapsto p_{(i,t)}(1|x)$ is not continuous !

Hypotheses

1) Lipschitz : There exists some $\gamma > 0$: *such that for all* z, z', n,

$$|\Phi(z,n) - \Phi(z',n)| \leq \gamma |z - z'|.$$

2) Uniform summability of the synaptic weights

$$\sup_i \sum_j |W_{j \to i}| < \infty.$$

3) Spontaneous spiking activity with intensity δ :

$$\Phi(\cdot, \cdot) \geq \delta > 0.$$

Theorem

Under the above hypotheses : If $\delta \geq \delta_*$, then

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1. *there exists a unique stationary chain* $X_t(i), t \in \mathbb{Z}, i \in \mathcal{I}$, *consistent with the dynamics.*
2. *the speed of convergence to equilibrium is bounded above :*

$$(2) \quad |E[f(X_s^t(i))|\mathcal{F}_0] - E[f(X_s^t(i))]| \leq C(t-s+1)\|f\|_\infty \varphi(s),$$

where $\varphi(s) \downarrow 0$ *as* $s \to \infty$.

Proof : Uses a conditional Kalikow-type decompositon in random environment (= spontaneous spikes) and comparison with a multitype branching process in random environment.

Back to neuroscience

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"indicating that a description of spiking as a stationary renewal process is a good approximation" (Gerstner and Kistler 2002).

In the same direction :

The statistical analysis of the spontaneous activity of several (but not all!) neurons in the hippocampus selects as best model a

renewal process.

– *Data registered by Sidarta Ribeiro (Brain Institute UFRN), in 2005.*

– *Data analyzed by Karina Y. Yaginuma, using the SMC.*

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Can we account for these apparently contradictory facts with our model ? Yes, we can ! ... and you will see how.

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Actually, our result is always true, but trivial and non-interesting if we are not working in the giant cluster.

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- Here, $p = \lambda / N$ and $\lambda = 1 + \vartheta / N$, $\vartheta > 0$.
- Observe that $W_{i \to j}$ and $W_{j \to i}$ are distinct and independent : being influenced by neuron i is different from influencing neuron i....

Does the past before the last spike of a neuron influence the future ?

/	/	/	/	/	/	/	/		1	0	0	0	0	0	?
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The point is : the last spike of neuron i before time L_t^i affects many neurons – different from i, which in turn affect other neurons and so on. How long does it take until this influence returns to the starting neuron i ?

This time is a sort of *recurrence time* in the random graph.

We can show :

Proposition

$$P(\textit{recurrence time} \leq k) \leq \frac{k}{N} e^{\vartheta k/N}.$$

$N =$ number of neurons.

$\vartheta =$parameter appearing in the definition of the synaptic weight probabilities, $Np = 1 + \vartheta/N$.

This implies

Theorem

On a "good set" of random synaptic weights :

$$|\textit{Covariance of neighboring inter-spike intervals}| \leq C\frac{1}{\delta^2}N(1-\delta)^{\sqrt{N}}.$$

Moreover,

$$P(\textit{good set}) \geq 1 - CN^{-1/2},$$

where δ *is the spontaneous spiking activity.*

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This conciliates the empirical results both of Goldberg et al. (1964) and of Nawrot et al. (2007) !

Thanks for your attention !

Proofs will be available in a few days in arXiv !